DISTRIBUTIONS WITH ZEROS AND POLES IN THE LAPLACE TRANSFORM OR GENERATING FUNCTION

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Dr. K.J. JOHN

INTRODUCTION

The theory of Laplace transforms has been extensively used in mathematical statistics, especially in connection with the study of properties of distributions. Even though the theory was originated by Laplace in the eighteenth century in connection with his studies on the power series technique for solving certain difference equations arising in the mathematical treatment of games of chance, its potential as a powerful tool in statistics was realized late until the discovery of the famous inversion theorem which helps one to uniquely determine the distribution. The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values, without the necessity of first finding the general solution and then evaluating from it the constants.

If f(x) is a single valued function of a real variable X, defined almost every where for $x \geq 0$ and is such that

$\int_0^\infty |f(x)|\, e^{-kx}\, dx$

Converges for some real value k, then f(x) is said to be Laplace transformable and

$\phi_f(s) = \int_0^\infty e^{-sx} f(x)\, dx$ ----------------- (1.1)

is the Laplace transform of f(x), where s = x + iy is a complex variable [Doetsch (1970)].

One Problem of special interest in probability theory is "under what conditions a rational Laplace transform corresponds to a probability density function?". This problem has been examined by several researchers. Lukacs and Szasz (1951,1952,1952a,1954) have developed some conditions in terms of Vandermonde determinants and non-negative trigonometric polynomials. Cox (1955) has discussed this question in the context of phase type distributions. Zemanian (1959,1960) has provided sufficient conditions using zeros and poles of Laplace transforms. Steutel (1967,1970) and Bondesson(1981) have studied the problem in the context of infinite divisibility.

Sumita and Masuda (1987) have considered Laplace transform with negative zeros and poles and have developed simple and useful conditions for Laplace transforms having single negative zeros and single negative poles to corresponds to a probability density function. Further they have identified some probability distributions having Laplace transforms with single negative zeros and single negative poles. It includes mixture of n independent exponential random variables and sums of n independent exponential random variables. Characterization theorems for these classes are given in terms of zeros and poles of the Laplace transform.

Motivated by this, in the present work we consider Laplace transforms having complex zeros and multiple negative poles. The class of probability density functions having Laplace transforms with complex zeros and multiple negative poles contains the mixture of n independent gamma variables and sums of n independent gamma variables. Characteristic properties of these distributions are obtained based on certain representations of the Laplace transform.

Most of the work in modeling statistical data using zeros and poles in the Laplace transform is centered around the continuous case. Only very little work seems to have been done in this direction in the discrete set up. When we turn our attention to the discrete set up the probability generating function comes up as a handy tool instead of Laplace transform. In the light of the above observation, we look in to the class of probability mass functions having probability generating function with positive poles and complex zeros. This class contains finite mixture of geometric and negative binomial distributions as well as their convolutions. Based on certain forms of representation of the generating functions, characterization theorems are obtained with regard to these models.

The present work is organized into five chapters of which after the present one, a review of some basic properties of Laplace transforms, which are required in the sequel, distributions having Laplace transforms with zeros and poles and properties and applications of probability generating functions are included in Chapter II.

We devote Chapter III to present some new results on distributions having Laplace transforms with complex zeros and negative poles. Situations where mixture of gamma distributions, which is a member of the above class, arise are examined and characterization theorems based on different forms of representation of its Laplace transform are derived. Further some results for convolution of gamma distributions are also presented in this chapter.

CHAPTER II

SOME PRELIMINARIES

In the present chapter we give a brief outline of the results that are of relevance in the succeeding chapters along with an outline of the important developments concerning probability distributions having Laplace transforms / generating functions with zeros and poles.

2.1 Some properties of Laplace Transforms.

The Laplace transform of a continuous single valued function of a real variable X , $0 \leq X < \infty$, defined by (1.1) has the property that the inverse Laplace transform can be expressed as an integral which can be evaluated using complex integration methods. In fact , if f(x) and f^I(x) are continuous functions on $x \geq 0$ and f(x) = 0 for x < 0 and if f(t) is O($e^{r_0 t}$) and ϕ_f(s) is given by

$$f(x) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{sx}\,\phi_f(s)\,ds \quad \text{---------------------------- (2.1)}$$

where s is complex and r is apositive constant.(Doetsch (1970)). In particular if F(.) is a distribution function for which the Laplace transform ϕ_f(s) exists for s > r then at all points of continuity , (Feller (1971)},

$$F(x) = \lim_{a\to\infty} \sum_{n \leq ax} \frac{(-a)^n}{n\,!}\phi^n(a) \quad \text{---------------------------- (2.2)}$$

If f(t) is a function that is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies

$$|\ f(t)\ | \leq M\, e^{t\gamma} \quad \text{------------------------------ (2.3)}$$

for all $t \geq 0$ and for some constants γ and M , then the Laplace transform of f(t) exists for all s >γ.

Another integral transform which has gained considerable interest is the Fourier transform is defined as

$$\Psi_f(t) = \int_{-\infty}^{\infty} e^{itx}\, f(x)\, dx \quad \text{------------------------------ (2.4)}$$

If f(x) is a probability density function, (2.4) turns out to be the characteristic function, which is fundamental in probability theory. The Laplace transform is a linear operator in the sense that for any two functions f(t) and g(t) whose Laplace transform exist,

$$\phi_{sf(t)+\,bg(t)}(s) = a\phi_{f(t)}(s) + b\,\phi_{g(t)}(s) \quad \text{--------------------------- (2.5)}$$

where a and b are constants.

If f_1(t) and f_2(t) satisfy the conditions of the existence theorem, then the product of their transforms ϕ_{f_1}(s) and ϕ_{f_2}(s) is the transform ϕ_f(s) of the convolution f(t) of f_1(t) and f_2(t) defined by

f(t) = $(f_1 * f_2)$(t)

= $\int_0^t f_1$(x) f_2(t-x) dx -------------------------- (2.6)

This represents the probability density function of the sum of two independent random variables with densities f_1(.) and f_2(.).

If f_1(.) , f_2(.) , . . . , f_n(t) corresponds to the probability density functions of independent random variables X1,X2, . . . Xn and if $f = f_1 * f_2 * \ldots * f_n$, then

ϕ_f(s) = $\prod_{i=1}^{n} \phi_{f_i}$(s) --------------------------- (2.7)

The Laplace transform ϕ_f(s) on [0 , ∞) is completely monotone if it possess derivatives ${\phi_f}^{(n)}$(s) with respect to s of all orders and

$(-1)^n {\phi_f}^{(n)}$(s) ≥ 0 --------------------------- (2.8)

The Bernstein's (1928) theorem states that the function defined on [0,∞) is the Laplace transform of a probability distribution f , if and only if it is completely monotone and $\phi_f(0)$ = 1.

A distribution F is infinitely divisible if for every n there exists a distribution Fn such that

F = $F_n^{n^*}$ ------------------ (2.9)

where$F_n^{n^*}$ is the convolution of n distribution functions.

The function ${\phi_f}^{(n)}$(s) is the Laplace transform of an infinitely divisible distribution if and only if it is of the form

W =$e^{-\Psi}$ ------------------- (2.10)

where

(λ) = $\int_0^\infty \frac{1 - e^{-\lambda x}}{x}$P{dx} -------------------- (2.11)

and P is a measure such that

$\int_0^\infty \frac{1}{x}$P{dx} <∞ -------------------- (2.12)

Key references concerning Laplace transforms and Fourier transforms are Widder (1946,1961) , Doetsch(1970) , Feller(1971) and Kreyszig(1993).

2.2. Probability distributions having Laplace transforms with negative zeros and poles.

The importance of characteristic functions as a powerful tool in the theory of probability is highlighted in the works of Lukacs (1970,1983) and Laha(1982). One of the problems of supreme importance in this study is that of obtaining necessary and sufficient conditions for complex valued function ϕ(t), of a real variable t, to be a characteristic function. The works of Khintchine(1937), Levy(1939), Cramer(1939), Polya(1949) provides affirmative answers to this problem.

Lukacs and Szasz (1951) have obtained a necessary condition that a polynomial without multiple root must satisfy in order that its reciprocal be a characteristic function. If ϕ(t) can be written in the in the form

$$\phi(t) = \{(1 - \frac{it}{v_1})(1 - \frac{it}{v_2}) \ldots (1 - \frac{it}{v_n})\}^{-1} \quad \text{------------------- (2.13)}$$

where v1,v2,. . . ,vn are complex numbers, the zeros of the polynomials are then given by

$t_j = -iv_j$, $j = 1,2,\ldots, n$.

For polynomials without multiple roots the zeros of the polynomials can be divided in to four groups.

I) Zeros $i\beta_h$, ($h= 1,2,\ldots,m$) on the positive imaginary axis($\beta_h > 0$);
II) Zeros $-i\alpha_j$, ($j= 1,2,\ldots,m$) on the negative imaginary axis($\alpha_j > 0$);
III) P symmetric pairs of complex roots in the upper half planes $iw_k$and $\overline{\iota\, w_k}$ where $w_k = c_k + id_k$, $c_k > 0$, $d_k > 0$ ($k=1,2,\ldots,p$)
IV) n symmetric pairs of complex roots in the lower half planes $-iv_m$ and $-i\overline{v_m}$
where $v_m = a_m + ib_m$, $a_m > 0$, $b_m > 0$, ($m = 1,2,\ldots,n$)

In the light of the above observations (2.13) can be written as

$$\phi(t)=\{\prod_{j=1}^{v}(1- \frac{it}{\alpha_j})\prod_{h=1}^{\mu}(1+ \frac{it}{\beta_h})\prod_{k=1}^{p}(1+ \frac{it}{w_k})(1- \frac{it}{\bar{w}_k})\prod_{m=1}^{n}(1- \frac{it}{v_m})(1- \frac{it}{v_n})\}^{-1} \quad \text{---------------- (2.14)}$$

Based on the representation (2.14), the following result is established in Lukacs and Szasz(1951).

Theorem 2.1

If the reciprocal of a polynomial without multiple roots is a characteristic function then the following conditions are satisfied.

a) The polynomial has no real roots. Its roots are located either all on the imaginary axis or in pairs$\pm$ b + ia, symmetric with regard to this axis.
b) If b + ia (a,b real ; a $\neq$ 0) is a root of the polinomial then it has at least one root iα such that sgnα = sgnα and $|\alpha| \leq |a|$.

Takano (1951) modified the above result deleting the assumption that the polynomial has no multiple roots. Moreover it is established that if the degree of the polynomial is less than 4, the condition is not only necessary but also sufficient. The results are summarized in the following theorem.

Theorem 2.2.

A necessary and sufficient condition that the complex valued function

$\phi(t)= \{c_0 + c_1(it) + c_2(it)^2 + . . . +c_n(it)^{n-1}\}^{-1}$ ----------------(2.15)

$c_n \neq 0$, of a real variable t for $n \leq 4$ to be a characteristic function is that

(i) $c_0 = 1$ and c_j are real

(ii) thepolynomial with real coefficients

$Q(z) = 1 + c_1z + c_2z^2 + . . . +c_nz^n$ has no pure imaginary roots and

(iii) If a$\pm$ ib (a ≠ 0, b ≠ 0) is a pair of complex roots of the polynomial Q(z) then it has

at least one root c such that sign c = sign a and $|c| \leq |a|$.

Lukacs and Szasz (1952) have proved that if a characteristic function $\phi(t)$ is analytic in a neighborhood of the origin, then it is analytic in a horizontal strip and can be represented in this strip by a Fourier integral where either this strip is in the whole plane or it has one or two horizontal boundary lines. The purely imaginary points on the boundary of the strip of convergence are singular points of $\phi(z)$. As a corollary to this result it is established that a necessary condition that a function, analytic in some neighborhood of the origin, be a characteristic function is that in either half plane the singularity nearest to the real axis be located on the imaginary axis.

Further it is established that an analytic characteristic function of an infinitely divisible distribution can have no zeros inside its strip of convergence. As a consequence of this it is established that an infinite divisible characteristic function can be obtained as the product of two non infinitely divisible characteristic functions. As an illustration they have considered

$$\phi(t)= \frac{\left(1+\frac{it}{v}\right)\left(1+\frac{it}{\bar{v}}\right)}{\left(1-\frac{it}{a}\right)\left(1-\frac{it}{v}\right)\left(1-\frac{it}{\bar{v}}\right)} \quad , \ v = a + ib. \qquad \text{------------(2.16)}$$

Then ϕ(t) is a characteristic function if

$$b \geq 2\sqrt{2}\, a \qquad \text{----------------------------------(2.17)}$$

Also ϕ(-t) is a characteristic function.

Now (t) = ϕ(t) ϕ(-t)

$$= \frac{1}{\left(1+\frac{t^2}{a^2}\right)}, \qquad \text{------------------------------(2.18)}$$

Which corresponds to the characteristic function of the Laplace distribution which is infinitely divisible. However ϕ(t) and ϕ(-t) are analytic characteristic function with zeros in the strip of convergence and hence they are not characteristic functions of infinitely divisible distributions.

In a subsequent paper Lukacs and Szasz (1952a) have improved upon Theorem (2.1) by imposing a restriction on the polynomial (2.13) . The result is reproduced as Theorem(2.3) below.

Theorem 2.3

The reciprocal of a polynomial whose roots are all single and have the same imaginary part is the Fourier transform of a distribution function if and only if

(i) The polynomial has one purely imaginary root ia (a ≠ 0) and n pairs of complex roots $\pm b_k + ia$, ($0 < b_1 < b_2 < . . . < b_n$, k = 1,2,...,n)

(ii) The determinant

$$g(\theta) = \begin{vmatrix} \text{Sin}^2\frac{b_1\theta}{2}, & \sin^2\frac{b_2\theta}{2}, & \ldots, & \sin^2\frac{b_n\theta}{2} \\ b_1^{\,2}, & b_2^{\,2}, & \ldots, & b_n^{\,2} \\ \ldots, & \ldots, & \ldots, & \ldots \\ b_1^{\,2(n-1)}, & b_2^{\,2(n-1)}, & \ldots, & b_n^{\,2(n-1)} \end{vmatrix} \geq 0 \quad \text{for all } \theta \quad \text{-------(2.19)}$$

Observing that the function g(ɵ) given by (2.19) is a cosine polynomial , situations where g(ɵ) is non negative for several configurations of the integers such as bj = i , bj = 2i-1, j = 1,2,...,n are also investigated.

Lukacs and Szasz (1954) defines the general Vandermonde determinant formed from the numbers ${b_1}^2,\ {b_2}^2,\ \ldots,\ {b_n}^2$ with exponents k, 1,2,...,(n-1) as

$$v_k = \begin{vmatrix} {b_1}^{2k}, & {b_2}^{2k}, & \ldots, & {b_n}^{2k} \\ {b_1}^{2}, & {b_2}^{2}, & \ldots, & {b_n}^{2} \\ \ldots, & \ldots, & \ldots, & \ldots \\ {b_1}^{2(n-1)}, & {b_2}^{2(n-1)}, & \ldots, & {b_n}^{2(n-1)} \end{vmatrix} \quad \text{-------(2.20)}$$

Denoting by g(ɵ), Vandermonde determinant formed from v_k with first raw replaced by $1 - \lambda_i \mathrm{Cos}\, b_i \theta$ with

$\lambda_j = \prod_{k=1}^{m}\left(1 - \frac{{b_j}^2}{{d_k}^2}\right)$, j = 1,2,...,n.

they obtained the following condition for a rational function to be a characteristic function.

Theorem 2.4.

The rational ϕ(t) is a characteristic function if and only if

$$g(\theta) = \begin{vmatrix} 1-\lambda_1 \mathrm{Cos}\, b_1\theta, & 1-\lambda_2 \mathrm{Cos}\, b_1\theta, & \ldots, & 1-\lambda_n \mathrm{Cos}\, b_1\theta \\ {b_1}^{2}, & {b_2}^{2}, & \ldots, & {b_n}^{2} \\ \ldots, & \ldots, & \ldots, & \ldots \\ {b_1}^{2(n-1)}, & {b_2}^{2(n-1)}, & \ldots, & {b_n}^{2(n-1)} \end{vmatrix} \geq 0 \quad \text{------(2.21)}$$

If the rational function has no zeros, that is, if ϕ(t) is the reciprocal of a polynomial whose roots are –ia, -ivj, -i$\overline{v_j}$, (j = 1,2,...,n), then the following theorem holds.

Theorem 2.5

The function ϕ(t) is a characteristic function if and only if

$$g(\theta) = \begin{vmatrix} 1\text{-Cos } b_1\theta, 1\text{-Cos } b_1\theta, \ldots, 1\text{-Cos } b_1\theta \\ {b_1}^2, \quad {b_2}^2, \quad \ldots, \quad {b_n}^2 \\ \ldots, \quad \ldots, \quad \ldots, \quad \ldots \\ {b_1}^{2(n-1)}, {b_2}^{2(n-1)}, \quad \ldots, \quad {b_n}^{2(n-1)} \end{vmatrix} \geq 0 \qquad \text{-------(2.22)}$$

For all ɵ.

Most of the investigations discussed above centers around zeros and poles in the characteristic function. Even though Fourier transform is defined for all bounded measures, it reduces to the characteristic function when the measures have unit mass. However the characteristic function of f(.) reduces to the Laplace transform $\phi_f(s)$, of f(.) when f is a probability density function on [0 , ∞) and s = - it . Hence it is usual to study the properties of distributions based on the functional form of Laplace transform . Cox (1955), in connection with his studies on the use of complex probabilities in stochastic process, consider probability density functions having Laplace transforms of the form

$$\phi_f(s) = \frac{(1+\frac{s}{\mu})}{(1+\frac{s}{\theta_1})(1+\frac{s}{\theta_2})} \qquad \text{----------------- (2.23)}$$

where$\mu, \theta_1$and θ_2 are real. Based on the above representation of $\phi_f(s)$ he has established the following result.

Theorem 2.6

The function $\phi_f(s)$, defined in (2.23) represents the Laplace transform of a probability distribution function if and only if

$\mu \geq \min(\theta_1,\theta_2)$ ---------------------- (2.24)

The investigations of Lukacs and Szasz discussed above centers around situations where all the poles and zeros in the characteristic function are simple and have the same real parts. Instead of using characteristic function ,Zemanian (1959,

1960) has studied Laplace transforms having multiple zeros and poles. Specifically they study probability distributions having Laplace transforms in the form

$$\phi_f(s) = \frac{\prod_{i=1}^{h}(s-\eta_i)\prod_{i=1}^{g}(s-v_i)}{\prod_{i=1}^{m}(s-\rho_i)\prod_{i=1}^{q}(s-\xi_i)} \text{----------------(2.25)}$$

where the η_i and ρ_i are real and v_i and ξ_i are complex. Here the corresponding probability density function is the sum of the exponentials except (possibly) for a constant. Moreover the complex poles occur in conjugate pairs , and all the poles will have negative real parts.

Steutel (1967) has examined infinite divisibility of exponential mixtures. For mixtures of exponential probability density functions in the form

$$f(x) = \sum_{j=1}^{n} a_j\lambda_j e^{-\lambda_j x} \qquad \text{------------------- (2.26)}$$

$a_j \neq 0$, $\sum_{j=1}^{n} a_j = 1$, and $\lambda_j>0$ with $0<\lambda_1<\lambda_2<. . . <\lambda_n$ the characteristic function simplyfies to

$$\phi_f(t) = \sum_{j=1}^{n}\left(\frac{a_j\lambda_j}{\lambda_j-it}\right)$$

$$= \frac{P(t)}{\prod_{j=1}^{n}(\lambda_j-it)} \text{--------------------(2.27)}$$

Where p(t) is a polynomial of degree not exceeding (n-1) . Using the above representation for $\phi_f(t)$, which cannot have more than (n-1) zeros, it is established that (2.26) is infinitely divisible if in the sequence {a1,a2, . . . ,an} there is no more than one change of sign.

In a subsequent work Steutel (1970) has also provided a simple and useful sufficient condition for a real valued function f(.) to be a probability density function based on the form of its Laplace transform. The result is given in the following theorem.

Theorem 2.7

Let f(.) be a real valued function on [0, ∞) such that

$$\phi_f(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{n}(1+\frac{s}{\theta_j})} \quad , \quad m > 0. \qquad \text{-------------(2.28)}$$

Then $[\theta_i<\eta_i,\ i = 1,2, . . . , m] \Rightarrow$ [f is a p.d.f. and is infinitely divisible]

Sumita and Masuda have studied the class of real functions on [0, ∞) having

Laplace transforms with only negative zeros and poles. If

$$\Omega = \left\{ f \setminus \phi_f(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{n}(1+\frac{s}{\theta_j})} \right. \qquad \text{-----------------(2.29)}$$

$0 \le m < n < \infty$, $\eta_i \neq \theta_j$, η_i , $\theta_j > 0$ for all i and j and Ω^+ is the class of all probability densities in Ω, then it is shown that Ω^+ is closed under convolution but not closed under mixing.

First up all they establish a necessary condition for a real valued function f to belong to Ω^+ which is given as Theorem 2.8 below.

Theorem 2.8

Let $f \in \Omega^+$ with $m > 0$ in (2.29) then $\theta_1 < \eta_1$.

After proving this necessary condition they proceed to seek necessary and sufficient conditions for Laplace transforms of the form (2.29) to correspond to a probability density function. In the special case when m = 1 and n= 2 in Theorem (2.8), it is established that the condition $\theta_1 < \eta_1$ is necessary and sufficient for f(.) to belong to Ω^+ . In case when m = 2, and n = 3 and all poles are of multiplicity one the following theorem is established.

Theorem 2.9

Let f ∈ Ω such that

$$\phi_f(s) = \frac{(1+\frac{s}{\eta_1})(1+\frac{s}{\eta_2})}{(1+\frac{s}{\theta_1})(1+\frac{s}{\theta_2})(1+\frac{s}{\theta_3})} \qquad \text{--------------(2.30)}$$

where $0 < \theta_1 < \theta_2 < \theta_3$. Then $f \in \Omega^+$ if and only if

(a) $\theta_1 < \eta_1$, $\theta_2 < \eta_2$.
Or
(b) $\theta_1 < \eta_1 \le \eta_2 < \theta_2 < \theta_3$ and

$$\frac{\{(\theta_3-\eta_1)(\theta_3-\eta_2)\}^{\theta_2-\theta_1}\{(\theta_1-\eta_1)(\theta_1-\eta_2)\}^{\theta_3-\theta_2}}{\{(\theta_2-\eta_1)(\theta_2-\eta_2)\}^{\theta_3-\theta_1}} \ge 1.$$

Observing that characterization of the class Ω^+ in terms of η_i *and* θ_j is a hard task for larger values of m and n, the case when m = 3 and n = 4 is examined and the relative positions of η_i *and* θ_j are located assuming $\theta_1 < \theta_2 < \theta_3 < \theta 4$.

The investigations of Sumita and Masuda (1987) center around probability densities having Laplace transforms with single negative zeros and single negative poles. However there may arise situations in which there can be multiple poles and multiple zeros. For instance the

Laplace transform corresponding to mixture of gamma densities posses multiple zeros and multiple poles. It is of special interest to examine properties of distributions having Laplace transforms with multiple zeros and poles, and this is undertaken in chapter III.

Sumita and Masuda (1987) further examine various classes of probability density functions in Ω^+. The class of finite mixture of exponential densities, denoted by CM_n, is defined as

CM_n= {f/ f(x) = $\sum_{i=1}^{n} p_i\theta_i e^{-\theta_i x}$, 0 < n <∞, θ_i , p_i> 0,
$\sum_{i=1}^{n} p_i$=1,$\theta_i \neq \theta_j$,fori≠j} ---------(2.31)
and the union of CMn is
CM $=\bigcup_{n=1}^{\infty} C$Mn
The class of probability density functions each of which is a finite convolution of exponential distribution is defined as
PF_n^* = {$f \setminus f = f_1 * f_2 * \ldots * f_n$, 0<n<∞, $f_i(x) = \theta_i \exp(-\theta_i x)$, θ_i>0, 1≤i ≤ n}
and

$$PF^* = \bigcup_{n=1}^{\infty} PF_n^*$$

It is established that both CM_n and PF_n^* are proper subsets of Ω^+ . Also based on the form of Laplace transform, the following characterization theorem is established.

Theorem 2.10

(a) [f ∈CM_n] ⇔ $\left[\phi_f(s) = \dfrac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{n}(1+\frac{s}{\theta_j})} \quad O<\theta_1<\eta_1 < \theta_2<\ldots < \theta_n \right]$

(b) [f ∈PF_n^*] ⇔ $\left[\phi_f(s) = \dfrac{1}{\prod_{j=1}^{n}(1+\frac{s}{\theta_j})}, \quad \theta_j > 0, \ I \leq j \leq n \right]$

(c) Both CM_n and PF_n^* are proper subsets of Ω^+ .

Another class of distributions under investigation is the class of probability density functions inCM_n or PF_n^* and probability density functions corresponding to sum of two independent random variables, one with a probability density function in CM_r and theother with a probability density functionin PF_l^* with n = r + l. This class, denoted by $CMPF_n$ is defined by

$CMPF_n = \{f \setminus f \in CM_n \cup . PF_n^* \text{ or } f = g * h,\ g \in CM_r,\ h \in PF_l^* \text{ and } n=r+l \}$

and

$$CMPF = \bigcup_{n=1}^{\infty} CMPF_n$$

A characterization theorem for the CMPFn class of distributions is reproduced as Theorem 2.11 below

Theorem 2.11

$$[f \in CMPF_n] \Leftrightarrow \left[f \in \Omega^+ \text{ with } \phi_f(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{n}(1+\frac{s}{\theta_j})} \right]$$

such thatO$<\theta_1<\eta_1 < \eta_2 < \ldots < \eta_m < \theta_n$, 0≤m<n and for i≤j≤n-1,

$I_j = (\theta_j, \theta_{j+1})$ contains at most one of $\eta_i's$.

Other classes of interest in Ω^+ are the classes of unimodal, strongly unimodal, log-concave probability density functions on [0,∞) and the class of probability density functions each of which is expressed as a finite convolution of probability density functions in CM_n , denoted by U, SU, LCC and SCM. They are defined as follows.

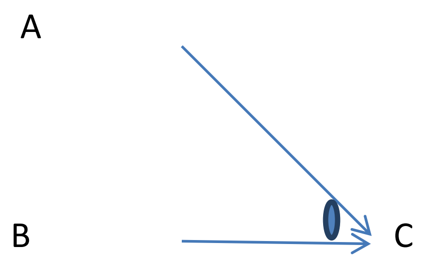
U = {f | there exists a real number x_0 such that f(x) ≤ f(y) for x ≤ y ≤ x_0 and f(x) ≥ f(y) for x0 ≤ x ≤ y}.

SU = {f | g ∈ U ⇒ f * g ∈ U, where the asterisk denotes convolution}

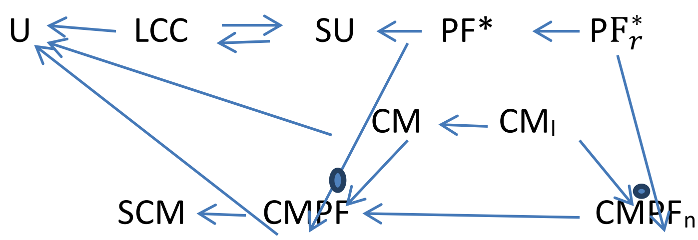
LCC = {f |log f(x) > 0} and there are no gaps in the interval of support}.

SCM = $\{f \setminus f = f_1 * f_2 * \ldots * f_n,\ f_i \in CM_{m_i},\ m_i > 0,\ 1 \le i \le n\}$.

Sumita and Masuda(1987) summarizes the relationship among the various classes of distributions using the following diagram. Here the arrow A→ B means that A is a proper subset of B. The symbol



means that A→B and B→C and f ∈ A, g ∈ B implies f*g ∈ C.



The relevance of the above classes of distributions in applied probability is also worth mentioning .Feller(1971) observes that for rational Laplace transform of the form

$$\phi(\lambda) = \frac{U(\lambda)}{V(\lambda)} \qquad \text{------------------ (2.33)}$$

where U(.) and V(.) are polynomials without common root and degree of U(.) is lower than that of V(.), if V(λ) = 0 has m distinct (real or imaginary) roots $\lambda_1, \lambda_2, \ldots, \lambda_m$, then ϕ(λ) admits the representation

$$\phi(\lambda) = \frac{\delta_1}{\lambda - \lambda_1} + \frac{\delta_2}{\lambda - \lambda_2} + \ldots + \frac{\delta_m}{\lambda - \lambda_m} \quad \text{-----------(2.34)}$$

The situation where such a representation admits multiple poles and zeros is of special interest and this aspect is investigated subsequently in Chapter 3. Many analytic functions belong to the class of meromorphic functions admit a partial fraction expansion of the form (2.34), the finite sum on the right being replaced by a uniformly convergent series.

In connection with busy periods and related branching process Feller(1971) consider Laplace transforms of a probability density function f(x) in the form

$\phi_f(s) = L(s + \lambda + - \lambda\beta(s))$ ------------------ (2.35)

where

$L(s) = \int_0^{\infty} e^{-sx}\, a(x)\, dx$

and

$\beta(s) = \int_0^{\infty} e^{-sx}\, b(x)\, dx$,

$\lambda > 0$ and both a(x) and b(x) are probability density functions. The expression (2.35) is used by Gaver (1962) and Keilson (1962) to study queues with service interruptions and priority queues.

2.3 Probability generating function.

For a discrete random variable X in the support of {0,1,2, …} and probability mass function f(x) , the probability generating function is defined as

$g_f(s) = \sum_{x=0}^{\infty} f(x)\, s^x$ ---------------------------- (2.36)

where $|s| \leq 1$.

Applications of probability generating function in the study of distributional properties is available in Feller (1968) and Johnson ,Kotz and Kemp (1992).

Abate and Whitt (1996) observes that for the special case of discrete distributions the Laplace – Stieltjes transform become probability generating function upon making the change of variable z = e^{-s} and that the generating function arises directly when we consider the moment generating function constructed from the power series representation of the Laplace Stieltjes transform

$\phi_f(s) = \sum_{k=0}^{\infty} \frac{m_k(-s)^k}{k!}$ with $m_0 = 1$.

In literature generating functions have received more attention than Laplace – Stieltjestransform , primarily because of their prominent role in combinatorics . The use of generating functions in combinatorics is extensively discussed in Wilf (1994).

2.4. Mixture distributions.

Finite mixture distributions have been extensively used in literature to model data in several branches of learning such as pattern recognition, remote sensing, mathematical geology etc. Smith (1985) has listed the following applications of mixture distributions.

(1) Fisheries research, where the k components (categories) are different ages.
(2) Sedimentology , where the categories are mineral types
(3) Medicine, where the categories are disease states.
(4) Economics, where the categories are discontinuous forms of behavior.

A systematic treatment of the structure of finite mixture distributions and the method of analysis of data from mixture distributions is given in Everitt and Hand (1981) and Smith (1985). Applications in speech analysis and in image analysis are given in Titterington (1990). There is also information about many special cases in the volume of Johnson and Kotz (1969,1970,1972).

Suppose that a random variable X takes values in a sample space $\mathcal{X}$ and that its distribution can be represented by a probability density function(or probability mass function in the case of discrete X) of the form

p(x) = $a_1 f_1$(x) + $a_2 f_2$(x) + …. + $a_k f_k$(x) ------------ (2.37)

x ∈ $\mathcal{X}$, a_j> 0, $a_1 + a_2 + \ldots + a_k = 1$, f_i(x) ≥ 0, $\int_x^{.} f_i(x)\,dx = 1$, j = 1,2, … ,k.

In such a case we say that X has a finite mixture distribution and p(.) defined by (2.37) is a finite mixture density function. a_1, a_2 , … , a_k are called mixing weights and f_1(.), f_2(.), … , f_k(.) are called the component densities of the mixture.

Mixtures of normal distribution is the focal theme of investigation in Elashoff (1972), Dalal (1978) and Ferguson (1983) . Mixtures of exponential distributions have important life testing applications .Nassar (1988) has used mixtures of exponential distributions in modeling life time data. For instance if a batch of electric bulbs consists of two subpopulations and the two subpopulations are in proportions p_1 and (1- p_1) and if the failure times of each population is assumed to be independently and exponentially distributed with mean $\frac{1}{\sigma_i}$, i = 1,2, the cumulative distribution function of the time to failure is

p_1[1 – exp($\frac{-t}{\sigma_1}$)] + (1 - p_1)[1 – exp($\frac{-t}{\sigma_2}$)]

which is a mixture of exponentials.

The cumulative distribution function of mixture of exponential distribution is given by

$$F(x) = 1 - \sum_{i=1}^{k} a_i e^{-\theta_i x} \qquad \text{-------------------} \quad (2.38)$$

$\theta_i > 0$, $a_j > 0$ and $\sum_{i=1}^{k} a_i = 1$.

For the distribution given in (2.38) the Laplace transform is

$$\phi_f(s) = \sum_{i=1}^{k} \left(\frac{a_i \theta_i}{s + \theta_i}\right) \qquad \text{------------------------------} \quad (2.39)$$

A k component mixture, in proportions $a_1, a_2, \ldots, a_k$, ($\sum_{i=1}^{k} a_i = 1$) of the Gamma distribution has probability density function

$$f(x) = \sum_{j=1}^{k} a_j \frac{\theta^{k+1-j}}{\Gamma(k+1-j)} x^{k-j} e^{-\theta x} \qquad \text{---------------} \quad (2.40)$$

$x \geq 0$, $\theta > 0$, $0 \leq a_j \leq 1$, $\sum_{j=1}^{k} a_j = 1$ and k is a positive integer.

A characterization of a mixture of gamma distribution using conditional moments is given by Adatia, Law and Wang (1991). Inference problems associated with a two component mixture of gamma distribution is given in Radhakrishna, Rao and Anjaneyulu (1992).

When the probability density function has the form (2.40) its Laplace transform turns out to be

$$\phi_f(s) = \sum_{j=1}^{k} \frac{a_j}{(1+\frac{s}{\theta})^{k+1-j}} \qquad \text{--------------------} \quad (2.41)$$

In Chapter III we look in to characterization problems associated with (2.40) using the form of their Laplace transform.

The concept of mixture distribution in the discrete case, introduced by Pearson (1915) , is physically motivated when the underlying random variable takes on numerical values. Identifiability of finite Poisson mixtures and finite binomial mixtures are examined by Teicher (1961) . Mixture of negative binomials , binomials and poisson are discussed in Rider (1962) . Asymptotic study of mixture distributions in the discrete set up is given in Johnson, Kotz and Kemp (1992).

In the light of definition (2.37) a finite mixture of geometric distribution has probability mass function

$$f(x) = \sum_{j=1}^{n} a_j p_j {q^x}_j \qquad \text{--------------------} \quad (2.42)$$

$0 \le p_j \le 1$, $q_j = 1 - p_j$, x = 0,1,2, … , $a_j > 0$ and $\sum_{j=1}^{k} a_j = 1$ and has the probability generating function

$$g_f(s) = \sum_{j=1}^{k} \left(\frac{a_j p_j}{1 - q_j s} \right) \text{--------------------} \quad (2.43)$$

A finite mixture of negative binomial distribution has the probability mass function specified by

$$f(x) = \sum_{j=1}^{n} a_j \binom{-j}{x} p^j (-q)^x \quad \text{--------------} \quad (2.44)$$

x = 0,1,2, … , $0 < n < \infty$, $0 \le p \le 1$, $q = 1 - p$, $a_j > 0$ and $\sum_{j=1}^{k} a_j = 1$

with probability generating function

$$g_f(s) = \sum_{j=1}^{n} \left(\frac{a_j p^j}{(1 - q_j s)^j} \right)$$

The expressions for the probability generating functions of the mixture of negative binomial and geometric distributions form the basis of characterization theorems in Chapter IV.

CHAPTER III

PROBABILITY DISTRIBUTIONS HAVING LAPLACE TRANSFORMS WITH ZEROS AND NEGATIVE POLES.

3.1 A class of probability density functions having Laplace transforms with zeros and negative poles

Following the terminology used in Chapter II, we denote by ϕ_f(s) the Laplace transform of a real function f(x). The present chapter deals with the probability density functions having rational Laplace transforms with zeros and negative poles.

We denote by C, the class of real functions f defined on [0, ∞) having Laplace transforms with complex zeros and negative poles. That is

$$C = \left\{ f \setminus \phi_f(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{n}(1+\frac{s}{\theta_j})} \right. \qquad \text{-----------(3.1)}$$

$O \leq \theta_1 \leq \theta_2 \leq. . . \leq \theta_n < \infty$; m and n are integers such that $0 \leq m < n < \infty$ and η_i's are complex.

Let C^+ denote the class of probability density functions in C. That is

$$C^+ = \{ f \setminus f \in C,\ f(x) \geq 0 \text{ and } \int_0^\infty f(x)dx = 1 \} \qquad \text{--------------- (3.2)}$$

Before looking into the properties of the class C^+ , we first consider the special case in which the θ_j' s and η_i's in (3.1) assume non negative real values. We discuss below some conditions on ϕ_f(s), so that ϕ_f(s) corresponds to a probability density function. Our first theorem deals with a possible range of s, in ϕ_f(s), that renders f a probability density function, which is given in K.J. John and K. R. M. Nair (1999).

Theorem 3.1

A necessary condition for the Laplace transform ϕ_f(s) , defined by

$$\phi_f(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{n}(1+\frac{s}{\theta_j})} \text{-----------}(3.3)$$

O $\leq \theta_1 \leq \theta_2 \leq . . . \leq \theta_n < \infty$, O $\leq \eta_1 \leq \eta_2 \leq . . . \leq \eta_m < \infty$, and m and n are integers such that $0 \leq$ m < n <∞ to correspond to a probability density function f(x); x ≥ 0 is that s > - θ_1.

Proof:

Using theorem 2.8, if f ∈ C^+ with m > 0, then $\theta_1 < \eta_1$.

If possible let s < - θ_1. Let e be such that

$\theta_1 <$ e $< \eta_1 < \theta_2$.

Then - $\eta_1 <$ -e < -θ_1

When s< - θ_1 , s can assume the value –e and therefore

$$\phi_f(-e) = \frac{\left(1-\frac{e}{\eta_1}\right)\left(1-\frac{e}{\eta_2}\right) \ldots (1-\frac{e}{\eta_m})}{\left(1-\frac{e}{\theta_1}\right)\left(1-\frac{e}{\theta_2}\right) \ldots (1-\frac{e}{\theta_n})}$$

Since all terms except the first term in the denominator in the above expression are positive, ϕ_f(-e)is negative. This contradicts the assumption that ϕ_f(s)is the Laplace transform of a probability density function. Hences > - θ_1.

Theorem 3.1 provides a lower bound for s in ϕ_f(s), the Laplace transform of f(.).

It may be noticed that the condition of the theorem is not sufficient. For instance if

$$\phi_f(s) = \frac{\left(1-\frac{s}{\frac{1}{5}}\right)}{\left(1-\frac{s}{\frac{1}{3}}\right)\left(1-\frac{s}{\frac{1}{2}}\right)}$$

the corresponding f(.) turns out to be

$$f(x) = -2 \,.\, \frac{1}{3} \,.\, e^{-\frac{1}{3}x} + 3 \,.\, \frac{1}{2} \,.\, e^{-\frac{1}{2}x}$$

$$= \frac{3}{2} \,.\, e^{-\frac{1}{2}x}\left(1 - \frac{4}{9} \,.\, e^{-\frac{x}{6}}\right)$$

which is not a probability density function.

Theorem 2.8 of Sumita and Masuda (1987) states that the condition

min $(\theta_1, \theta_2, \ . \ . \ . \ , \theta_n) \leq$ min $(\eta_1 \ , \eta_2 \ , \ . \ . \ . \ . \ , \eta_m)$

is necessary for ϕ_f(s) to be a Laplace transform.

However if

$$\phi_f(s) = \frac{\left(1-\frac{s}{2.5}\right)\left(1-\frac{s}{3}\right)\left(1-\frac{s}{3.5}\right)}{\left(1-\frac{s}{2}\right)\left(1-\frac{s}{4}\right)\left(1-\frac{s}{5}\right)\left(1-\frac{s}{6}\right)}$$

Then $\theta_1 = 2 < 2.5 = \eta_1$. But ϕ_f(s) is not the Laplace transform of a probability density function. Hence the condition of the theorem is not sufficient.

In the light of the above observation we relax the condition of the theorem, choosing $\theta_1 = \theta_2 = \ . \ . \ .= \theta_n$, in which case ϕ_f(s) has a multiple pole of order n, so as to get a necessary and sufficient condition for ϕ_f(s) in (3.3) tocorrespond to the Laplace transform of a probability density function, which is given in K.J. John and K. R. M. Nair (1999).

Theorem 3.2

Assume that f(.) is a real valued functionover [0, ∞)for which the Laplace transform has the form

$$\phi_f(s) = \frac{\prod_{i=1}^{m}\left(1+\frac{s}{\eta_i}\right)}{\left(1+\frac{s}{\theta}\right)^n} \text{----------------} \quad (3.4)$$

$O < \eta_1 \leq \eta_2 \leq . . . \leq \eta_m < \infty$, $0 < \theta < \infty$, m and n are positive integers such that m < n, n > 1. Then $\phi_f(s)$ represents the Laplace transform of a probability density function if and only if $\theta < \eta_1$.

Proof :

Assume that (3.4) holds. Taking logarithm of both sides of (3.4) and differentiating with respect to s we get

$\phi^I{}_f(s) = \phi_f(s)\Psi(s)$ ----------------(3.5)

Where $\phi^I{}_f(s) = \frac{d\phi_f(s)}{ds}$ and

$\Psi(s) = \sum_{i=1}^{m} \frac{1}{\eta_i + s} - \frac{n}{s+\theta}$

Notice that $\Psi(s)$ is negative if $\theta < \eta_1$. Successive differentiation of (3.5) gives

$$\phi^{(k)}{}_f(s) = \phi^{(k-1)}{}_f(s)\Psi(s) + \binom{k-1}{1} \phi^{(k-2)}{}_f(s)\Psi^{(1)}(s) + \ldots +$$

$$\binom{k-1}{i} \phi^{(k-i-1)}{}_f(s)\Psi^{(i)}(s) + \ldots + \phi_f(s)\Psi^{(k-1)}(s) \text{ ------------(3.6)}$$

Where $\phi^{(i)}{}_f(s) = \frac{d^i\phi_f(s)}{ds^i}$, i = 1,2, . . . , k.

and $\Psi^{(i)}(s) = \frac{d^i\Psi(s)}{ds^i}$, i = 1,2, . . . , k-1.

Observing that

$\Psi^{(k)}(s) < 0$ for k = 0,2,4, …

And

$\Psi^{(k)}(s) > 0$ for k = 1,3,5,…

We have

$\phi^{(k)}{}_f(s) > 0$ for k = 2,4,6,…

And

$\phi^{(k)}{}_f(s) < 0$ for $k = 1,3,5,\ldots$

From (3.6) we conclude that

$(-1)^k\phi^{(k)}{}_f(s) \geq 0$ for all integers k. ---------------- (3.7)

It may be noticed from (3.7) that $\phi_f(s)$ is completely monotone. Using Bernstein's theorem, in Section(2.1), we see that $\phi_f(s)$ specified by (3.4) is the Laplace transform of a probability density function.

The converse follows from Sumita and Masuda(1987).

It may be noticed that Theorem 3.2 includes the situation when the Laplace transform has multiple negative zeros and a single multiple negative pole, since as per conditions of the theorem, some of the η_i's may be equal. However the situations when the Laplace transforms has multiple negative zeros and multiple negative poles is not covered by the theorem.

3.2 Mixtures of gamma distributions

In this section we consider classes of probability density functions in C^+, the main emphasis being on mixtures of gamma densities is specified by the probability density function

$$f(x) = \sum_{j=1}^{n} a_j \frac{\theta^{n+1-j}}{\Gamma(n+1-j)} x^{n-j} e^{-\theta x} \quad \text{---------------} \quad (3.8)$$

$x \geq 0$, $\theta > 0$, $0 \leq a_j \leq 1$, $\sum_{j=1}^{n} a_j = 1$ and n is a positive integer.

Such densities arises naturally in various contexts. For example consider N independent and identically distributed random variables $X_1, X_2, \ldots, X_N$ with a common distribution function F(.) and Laplace transform $\phi(s)$. Define

$S_N = \sum_{i=1}^{N} X_i$.

Take N as a random variable independent of the X_i's with

P(N = j) a_{n+1-j}, j = 1,2, … , n.

For any j , the Laplace transform of S_j is then

$$\phi_j(s) = E(e^{-sS_j})$$

$$= [\phi(s)]^j \ .$$

Now the Laplace transform of S_N is

$$\phi_{S_N}(s) = E(e^{-sS_N})$$

$$= E_N\, E(e^{-sS_N} / N=j)$$

$$= \sum_{j=1}^{n} E(e^{-sS_j}) P(N=j)$$

Since N is independent of the X_i's,

$$\phi_{S_N}(s) = \sum_{j=1}^{n} [\phi(s)]^j a_{n+1-j}$$

Specializing to the case when Xi's are exponential random variables with common distribution function

$$F(x) = 1 - e^{-\theta x}; \ x > 0, \ \theta > 0,$$

we get

$$\phi_{S_N}(s) = \sum_{j=1}^{n} (1 + \frac{s}{\theta})^{-j} a_{n+1-j} \text{-------------} \quad (3.9)$$

Since $(1 + \frac{s}{\theta})^{-j}$ is the Laplace transform of a gamma variable with scale parameter θ and shape parameter j , by inversion theorem the distribution of S_N turns out to be (3.8)

Finite mixture of gamma densities defined in (3.8) can also be considered as a generalized length biased distribution . Denoting A_{n-1}(x), a polynomial of degree (n-1) having a stipulated form and h(x) the probability density function of a non negative continuous random variable X, Lingappaiah (1988) defines the length based distribution as

f(x) = A_{n-1}(x) h(x) ------------------- (3.10)

Taking

$$A_{n-1}(x) = a_1 \frac{x^{n-1}}{\mu^I{}_{n-1}} + a_2 \frac{x^{n-2}}{\mu^I{}_{n-2}} + \ldots + a_n \frac{1}{\mu^I{}_0}$$

With $a_i > 0$, $\sum_{i=1}^{n} a_i = 1$,$\mu^I{}_i$ = E(X^i) and X is a random variable having probability density function

$$H(x) = \theta e^{-\theta x}, \; x > 0, \theta > 0,$$

so that

$$\mu^I{}_i = \frac{r\,!}{\theta^r},$$

the function f(x) defined in (3.10) reducesto a finite mixture of gamma densities given by (3.8). The following theorem characterizes the probability distribution specified by (3.8) in terms of the form of the Laplace transform.

Theorem 3.3

Let X be a continuous non negative random variable with a probability density function f(x). Then X follows the mixture of gamma distribution specified by (3.8) if and only if its Laplace transform has the representation

$$\phi_f(s) = \frac{\prod_{i=1}^{n-1}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta})^n} \text{---------------- } (3.11)$$

where - η_i's are the roots of the polynomial equation

$\sum_{j=1}^{n}(1+\frac{s}{\theta})^{-j}\, a_j$------------- (3.12)

Proof:

When X has probability density function (3.8) its Laplace transform simplifies to

$$\phi_f(s) = \frac{a_1}{(1+\frac{s}{\theta})^n} + \frac{a_2}{(1+\frac{s}{\theta})^{n-1}} + \dots + \frac{a_n}{(1+\frac{s}{\theta})}.$$

$$= \frac{a_1+a_2\left(1+\frac{s}{\theta}\right)+\dots+a_n\left(1+\frac{s}{\theta}\right)^{n-1}}{(1+\frac{s}{\theta})^n}$$

$$= \frac{\sum_{j=1}^{n}.a_j\left(1+\frac{s}{\theta}\right)^{j-1}}{(1+\frac{s}{\theta})^n}$$

$$= \frac{c\ \prod_{i=1}^{n-1}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta})^n} \text{-----------------(3.13)}$$

where η_i's are given by (3.12). The initial condition $\phi_f(0) = 1$ gives c = 1, so that (3.13) takes the form

$$\phi_f(s) = \frac{\prod_{i=1}^{n-1}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta})^n}$$

as claimed . Conversely when $\phi_f(s)$ has the form (3.11), using (3.12) we can write

$$\phi_f(s) = \frac{\sum_{j=1}^{n}.a_j\left(1+\frac{s}{\theta}\right)^{j-1}}{(1+\frac{s}{\theta})^n}$$

$$= \frac{a_1+a_2\left(1+\frac{s}{\theta}\right)+\cdots+a_n\left(1+\frac{s}{\theta}\right)^{n-1}}{(1+\frac{s}{\theta})^n}$$

$$= \frac{a_1}{(1+\frac{s}{\theta})^n} + \frac{a_2}{(1+\frac{s}{\theta})^{n-1}} + \ldots + \frac{a_n}{(1+\frac{s}{\theta})}.$$

whichis the Laplace transform of (3.8). Therefore

$$f(x) = \sum_{j=1}^{n} a_j \frac{\theta^{n+1-j}}{\Gamma(n+1-j)} x^{n-j} e^{-\theta x}, \quad x \geq 0, \ \theta > 0,$$

$0 \leq a_j \leq 1$, $\sum_{j=1}^{n} a_j = 1$ and n is a positive integer.

In particular when n = 2 in (3.11) , the Laplace transform takes the form

$$\phi_f(s) = \frac{(1+\frac{s}{\eta_1})}{(1+\frac{s}{\theta})^2}, \quad 0<\theta<\infty, \ 0<\eta_1<\infty.$$

This is the Laplace transform of a probability density function if - η_i is the solution of the polynomial

$$\sum_{j=1}^{2}. a_j \left(1+\frac{s}{\theta}\right)^{j-1} = 0.$$

That is when

$$\sum_{j=1}^{2}. a_j \left(1+\frac{s}{\theta}\right)^{j-1} = 1 + \frac{s}{\eta_1}$$

where $a_j > 0$ for j = 1,2 and $\sum_{j=1}^{2}. a_j = 1.$ Equating the coefficients of s in the above expression we get $\eta_1 a_2 = \theta$. Since $0 < a_2 < 1$ we have $\theta < \eta_1$. This condition is the same as that of Cox (1955) [Theorem 2.6], when $\theta_1 = \theta_2 = \theta$.

Theorem 3.3 provides a necessary and sufficient condition for a gamma mixture to have Laplace transform with zeros and a multiple negative pole. This enables one to determine the distribution uniquely through a knowledge of the pole and zeros in the Laplace transform. We give below two examples by way of illustration. Assume that $\phi_f(s)$ has a multiple pole and single zeros in the form

$$\phi_f(s) = \frac{\left(1-\frac{s}{2}\right)\left(1-\frac{s}{3}\right)}{(1-s)^3}.$$

Since $\phi_f(s)$ can be represented as

$$\phi_f(s) = \frac{\frac{1}{6}}{(1+s)} + \frac{\frac{1}{2}}{(1+s)^2} + \frac{\frac{1}{3}}{(1+s)^3}.$$

the corresponding probability density function is given by

$$f(x) = \frac{1}{6}e^{-x} + \frac{1}{2}\frac{x}{\Gamma(2)}e^{-x} + \frac{1}{3}\frac{x^2}{\Gamma(3)}e^{-x}\ ; \ x > 0$$

which is the probability density function of a mixture of gamma distributions.

Next we consider a situation when the Laplace transform is having a multiple pole and multiple zeros. The Laplace transform

$$\phi_f(s) = \frac{\left(1+\frac{s}{4}\right)^2}{\left(1+\frac{s}{2}\right)^3}$$

canbe written in the form

$$\phi_f(s) = \frac{\frac{1}{4}}{(1+\frac{s}{2})} + \frac{\frac{1}{2}}{(1+\frac{s}{2})^2} + \frac{\frac{1}{4}}{(1+\frac{s}{2})^3}.$$

from which we get the form of the probability density function as

$$f(x) = \frac{1}{2}e^{-2x} + 2\frac{x}{\Gamma(2)}e^{-2x} + 2\frac{x^2}{\Gamma(3)}e^{-2x}\ ; \ x > 0$$

which is a mixture of gamma densities.

A variant of (3.8) can be obtained by changing the number of terms in the summand. The gamma mixture under investigation is specified by the probability density function

$$f(x) = \sum_{j=1}^{m} a_j \frac{\theta^{n+1-j}}{\Gamma(n+1-j)} x^{n-j} e^{-\theta x} \quad , \text{ ---------------(3.14)}$$

$x \geq 0$, $\theta > 0$, m and n are positive integers with m = n-k, k = 1,2,3, … , (n-1), $0 \leq a_j \leq 1$, $\sum_{j=1}^{m} a_j = 1$. It may be noticed that (3.4) reduces to (3.8) when m = n. Theorem 3.3 can be modified for the class of densities defined by (3.14) as follows.

Theorem 3.4

The probability density function of a continuous non negative random variable X has the form (3.14) if and only if its Laplace transform has the representation

$$\phi_f(s) = \frac{\prod_{i=1}^{m-1}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta})^n} \text{ ---------------- (3.15)}$$

$0 < \theta < \infty$, m and n are positive integers such that m < n and $-\eta_i$'s are the roots of the polynomial

$$\sum_{j=1}^{m} . a_j \left(1+\frac{s}{\theta}\right)^{j-1} = 0. \text{------------(3.16)}$$

Where $a_j > 0$ for j = 1,2. … , m and $\sum_{j=1}^{m} . a_j = 1$.

The theorem follows by proceeding along the same lines as in the proof of Theorem 3.3 by representing the Laplace transform of (3.14) in the form (3.15) utilizing (3.16)

A special case of (3.14) , by imposing a restriction on the mixing probabilities, is

f(x) = $\sum_{j=1}^{m} a_j \frac{\theta^{n+1-j}}{\Gamma(n+1-j)} x^{n-j} e^{-\theta x}$, ---------------(3.14)

x≥ 0, θ> 0, m and n are positive integers with m ≤ n , $\sum_{j=1}^{m} a_j$ = 1, a_j> 0

and

$\sum_{j=1}^{m} a_j\, y^{j-1}$= 0 ----------------- (3.18)

has negative real roots < - θ .

We give below a characterization of (3.170 in terms of the Laplace transform. The result is published in John (1991).

Theorem 3.5

The probability density function of a non negative random variable X is of the form (3.17) if and only if its Laplace transform is given by

$$\phi_f(\text{s}) = \frac{\prod_{i=1}^{m-1}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta})^n} \text{------------ (3.19)}$$

0 ≤ m-1 < n, 0 < $\theta < \eta_1 \leq \eta_2 \leq \ldots \leq \eta_{m-1}$<∞ .

Proof:

From the representation of (3.17),

f(x) = $a_1 \frac{\theta^n}{\Gamma(n)} x^{n-1} e^{-\theta x}$ + $a_2 \frac{\theta^{n-1}}{\Gamma(n-1)} x^{n-2} e^{-\theta x}$ + … + $a_m \frac{\theta^{n-m+1}}{\Gamma(n-m+1)} x^{n-m} e^{-\theta x}$.

Its Laplace transform is given by

$$\phi_f(s) = \frac{a_1}{(1+\frac{s}{\theta})^n} + \frac{a_2}{(1+\frac{s}{\theta})^{n-1}} + \ldots + \frac{a_m}{(1+\frac{s}{\theta})^{\,n-m+1}}.$$

$$= \frac{a_1+a_2\left(1+\frac{s}{\theta}\right)+\cdots+a_m\left(1+\frac{s}{\theta}\right)^{m-1}}{(1+\frac{s}{\theta})^n}$$

$$= \frac{\sum_{j=1}^{m} a_j\left(1+\frac{s}{\theta}\right)^{j-1}}{(1+\frac{s}{\theta})^n}$$

$$= \frac{c\ \prod_{i=1}^{m-1}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta})^n}$$

where η_i's satisfy (3.18) .

But $\phi_f(0) = 1$. This gives c = 1.

Therefore from (3.20), $\phi_f(s)$ has the form (3.9).

Conversely let $\phi_f(s)$ takes the form (3.19). Then we can write

$$\phi_f(s) = \frac{A_1}{(1+\frac{s}{\theta})^n} + \frac{A_2}{(1+\frac{s}{\theta})^{n-1}} + \ldots + \frac{A_m}{(1+\frac{s}{\theta})^{\,n-m+1}}.$$

which gives

$$f(x) = \sum_{j=1}^{m} A_j \frac{\theta^{n+1-j}}{\Gamma(n+1-j)} x^{n-j} e^{-\theta x}$$

$0 \le x < \infty$, $0 < n < \infty$, $\theta > 0$, $m \le n$, , $\sum_{j=1}^{m} A_j = 1$ and , $\sum_{j=1}^{m} A_j\, y^{j-1} = 0$ has negative real roots $< -\theta$.

Generalization of the mixture of gamma distribution defined in (3.8) in the form (3.14) admits only one multiple pole in the Laplace transform. In the sequel, we improve upon the form of the probability density function(3.8) in such a way that the Laplace transform admits several multiple poles.

Let X be a continuous non negative random variable with probability density function

$$f(x) = \sum_{i=1}^{2} . \sum_{j=1}^{n_i} a_{ij} \frac{{\theta_i}^{n_i+1-j}}{\Gamma(n_i+1-j)} x^{n_i-j} e^{-\theta_i x} \text{------(3.21)}$$

$0 < n_i < \infty$, i = 1,2, θ_1 , $\theta_2 > 0$, x > 0, $0 \le a_{ij} \le 1$ and $\sum_{i=1}^{2} . \sum_{j=1}^{n_i} a_{ij} = 1$.

Next theorem provides a characterization of (3.21) based on the form of the Laplace transform.

Theorem 3.6

A probability density function f(x) of a non negativecontinuous random variable X takes the form (3.21) , if and only if its Laplace transform has the representation

$$\phi_f(s) = \frac{\prod_{i=1}^{n_1+n_2-1}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta_1})^{n_1}(1+\frac{s}{\theta_2})^{n_2}} \text{------------------------- (3.22)}$$

wheren_1 and n_2 are positive integers , $0 < \theta_i < \infty$, i = 1,2 and - η_i's are the roots of the polynomial equation

$$(1+\frac{s}{\theta_2})^{n_2} \sum_{i=1}^{n_1} . a_{1i} \left(1+\frac{s}{\theta_1}\right)^{n_1-i-1} + (1+\frac{s}{\theta_1})^{n_1} \sum_{j=1}^{n_2} . a_{2j} \left(1+\frac{s}{\theta_2}\right)^{n_2-j-1} = 0$$

with $0 \le a_{ij} \le 1$ and $\sum_{i=1}^{2} . \sum_{j=1}^{n_i} a_{ij} = 1$.

Proof:

Let X has a probability density function(3.21). Its Laplace transform is given by

$$\phi_f(s) = \frac{a_{11}}{(1+\frac{s}{\theta_1})^{n_1}} + \frac{a_{12}}{(1+\frac{s}{\theta_1})^{n_1-1}} + \dots + \frac{a_{1n_1}}{(1+\frac{s}{\theta_1})} \cdot$$

$$+ \quad \frac{a_{21}}{(1+\frac{s}{\theta_2})^{n_2}} + \frac{a_{22}}{(1+\frac{s}{\theta_2})^{n_2-1}} + \ \ldots \ + \frac{a_{2n_2}}{(1+\frac{s}{\theta_2})}.$$

$$= \frac{(1+\frac{s}{\theta_2})^{n_2} \sum_{i=1}^{n_1} a_{1i}\left(1+\frac{s}{\theta_1}\right)^{n_1-i-1} + (1+\frac{s}{\theta_1})^{n_1} \sum_{j=1}^{n_2} a_{2j}\left(1+\frac{s}{\theta_2}\right)^{n_2-j-1}}{(1+\frac{s}{\theta_1})^{n_1}(1+\frac{s}{\theta_2})^{n_2}}$$

$$= \frac{c \ \prod_{i=1}^{n_1+n_2-1}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta_1})^{n_1}(1+\frac{s}{\theta_2})^{n_2}} \text{----------------- (3.24)}$$

using (3.23).

But $\phi_f(0) = 1$. Therefore from (3.24), c = 1 and hence we get the form of $\phi_f(s)$as in (3.22).

Conversely when $\phi_f(s)$ takes the form (3.22),

$$\phi_f(s) = \frac{(1+\frac{s}{\theta_2})^{n_2} \sum_{i=1}^{n_1} a_{1i}\left(1+\frac{s}{\theta_1}\right)^{n_1-i-1} + (1+\frac{s}{\theta_1})^{n_1} \sum_{j=1}^{n_2} a_{2j}\left(1+\frac{s}{\theta_2}\right)^{n_2-j-1}}{(1+\frac{s}{\theta_1})^{n_1}(1+\frac{s}{\theta_2})^{n_2}}$$

$$= \quad \frac{a_{11}}{(1+\frac{s}{\theta_1})^{n_1}} + \frac{a_{12}}{(1+\frac{s}{\theta_1})^{n_1-1}} + \ \ldots \ + \frac{a_{1n_1}}{(1+\frac{s}{\theta_1})}.$$

$$+ \quad \frac{a_{21}}{(1+\frac{s}{\theta_2})^{n_2}} + \frac{a_{22}}{(1+\frac{s}{\theta_2})^{n_2-1}} + \ \ldots \ + \frac{a_{2n_2}}{(1+\frac{s}{\theta_2})}. \text{-----(3.25)}$$

From(3.25) we get f(x) in the form (3.21).

When the Laplace transform admits k multiple poles the natural extension of (3.21) takes the form

$$f(x) = \sum_{i=1}^{k}.\sum_{j=1}^{n_i} a_{ij} \frac{{\theta_i}^{n_i+1-j}}{\Gamma(n_i+1-j)} x^{n_i-j} e^{-\theta_i x} \quad \text{------(3.21)}$$

$x > 0$, $0 < n_i < \infty$, $\theta_i > 0$, $i = 1,2, \ldots, k$, $0 \leq a_{ij} \leq 1$ and $\sum_{i=1}^{k}.\sum_{j=1}^{n_i} a_{ij} = 1$.

Analogous to theorem(3.6), a characterization for the above class of densities can be formulated as follows.

Theorem 3.7

The probability density function of a non negative continuous random variable X has the form (3.26) if and only if its Laplace transform has the representation

$$\phi_f(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{(1+\frac{s}{\theta_1})^{n_1}(1+\frac{s}{\theta_2})^{n_2}\ldots(1+\frac{s}{\theta_k})^{n_k}} \quad \text{---------------------- (3.27)}$$

where $m = \sum_{i=1}^{k} n_i - 1$, $0 < \theta_i < \infty$ for I = 1,2, … , k and $-\eta_i$'s are the roots of the equation

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{ij} \left(1+\frac{s}{\theta_i}\right)^{j-1} \prod_{\substack{z=1 \\ z\neq i}}^{k}(1+\frac{s}{\theta_z})^{n_z} = 0 \quad \text{------------- (3.28)}$$

with $0 \leq a_{ij} \leq 1$ and $\sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{ij} = 1$.

The theorem follows by finding the Laplace transform of (3.26) subject to condition (3.28) and the converse follows by expressing $\phi_f(s)$ in the expanded form analogous to the proof of Theorem (3.6)

3.3 Convolution of a mixture of gamma densities.

An interesting property of the gamma distribution is that it is closed under convolutions. For the gamma distribution with shape parameter p and scale parameter m, the Laplace transform turns out to be

$$\phi_f(s) = (1 + \frac{s}{m})^{-p}.$$

The Laplace transform of the convolution of k gamma densities with shape parameters $p_1, p_2, \ldots, p_k$, simplies to

$$\phi_{f_1 * f_2 * \ldots * f_k} = (1 + \frac{s}{m})^{-(p_1 + p_2 + \ldots + p_k)}$$

which is the Laplace transform of a gamma distribution with shape parameter $(p_1 + p_2 + \ldots + p_k)$.

A finite convolution of mixture of gamma densities is defined as

$$f = f_1 * f_2 * \ldots * f_k \text{----------------} \quad (3.29)$$

where

$$f_i(x) = \sum_{j=1}^{m_i} a_j \frac{{\theta_i}^{n_i+1-j}}{\Gamma(n_i+1-j)} x^{n_i-j} e^{-\theta_i x} \text{--------}(3.30)$$

$$x > 0,\ 0 < n_i < \infty,\ m_i \le n_i,\ \theta_i > 0,\ \sum_{j=1}^{n} a_j = 1.$$

Instead of utilizing (3.29) , we consider a special case of (3.29), obtained by imposing some restrictions on the parameters.

(a) Assume that the equation

$\sum_{j=1}^{m} a_j\, y^{j-1} = 0$ has negative real roots $< -\theta_i$ and

(b) the scale parameter θ_i's are such that

$0 < \theta_1 < \theta_2 < \ldots < \theta_k < \infty$

and the interval $I_j = (\theta_j, \theta_{j+1})$, j = 1,2, … , k contains at most one zero of $\phi_{f_i}(s)$ defined in (3.1)

The following theorem provides a characterization of convolution of gamma mixture in terms of the Laplace transforms having negative zeros and negative poles in the above set up.

Theorem 3.8.

The probability density function f(x) of a non negative continuous random variable has the form (3.29), with parameters satisfying (a) and (b) if and only if the Laplace transform can be expressed in the form

$$\phi_f(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{k}(1+\frac{s}{\theta_j})^{n_j}} \text{-------------} (3.31)$$

$0 < \theta_1 < \eta_1 < \theta_2 < \eta_2 < \ldots < \theta_k < \eta_k \leq \eta_{k+1} \leq \ldots \leq \eta_m < \infty$ and

$0 \leq m = \sum_{i=1}^{k}(m_i - 1) < \sum_{j=1}^{k} n_j$.

Proof :

Let $f = f_1 * f_2 * \ldots * f_k$ where $f_i(x)$ is of the form (3.30). In the light of Theorem (3.5) we have

$$\phi_{f_i}(s) = \frac{\prod_{j=1}^{m_i-1}(1+\frac{s}{\eta_{ij}})}{(1+\frac{s}{\theta_i})^{n_i}} \text{------------------} (3.32)$$

$0 \leq m_i - 1 < n_i$, $0 < \theta_i < \lambda_{i1} \leq \lambda_{i2} \leq \ldots \leq \lambda_{i\, m_i-1} < \infty$.

Now

$$\phi_f(s) = \prod_{i=1}^{k} \phi_{f_i}(s)$$

$$= \prod_{i=1}^{k} \left(\frac{\prod_{j=1}^{m_i-1}(1+\frac{s}{\eta_{ij}})}{(1+\frac{s}{\theta_i})^{n_i}} \right)$$

since

$\eta_i \in \{\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1\,m_1-1}, \lambda_{21}, \lambda_{22}, \ldots, \lambda_{2\,m_2-1}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{k\,m_k-1}\}$

for i =1,2, . . . , m, we can write $\phi_f(s)$as

$$\phi_f(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{k}(1+\frac{s}{\theta_j})^{n_j}}$$

where $0 \leq m = \sum_{i=1}^{k}(m_i - 1) < \sum_{j=1}^{k} n_j$. The condition

$0 < \theta_1 < \eta_1 < \theta_2 < \eta_2 < \ldots < \theta_k < \eta_k \leq \eta_{k+1} \leq \ldots \leq \eta_m < \infty$

follows from (b).

The numerator of (3.31) can be rearranged by considering those

$\lambda_{ij} \in \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, j = 1,2, . . ., $m_i - 1$

which are greater than θ_i, i = 1,2,…, k and then taking the product so that

$$\phi_f(s) = \frac{\prod_{i=1}^{k} \cdot \prod_{j=1}^{m_i-1}(1+\frac{s}{\lambda_{ij}})}{\prod_{i=1}^{k}(1+\frac{s}{\theta_i})^{n_i}}$$

$0 \leq m_i - 1 < n_i$, $\theta_i < \lambda_{ij}$, j = 1,2, … , $m_i - 1$

$$= \prod_{i=1}^{k} \left(\frac{\prod_{j=1}^{m_i-1}(1+\frac{s}{\eta_{ij}})}{(1+\frac{s}{\theta_i})^{n_i}} \right)$$

$$= \prod_{i=1}^{k} \phi_{f_i}(s) \qquad \text{by (3.32)}$$

By (2.7), f(x) will take the form (3.29).

In Theorem (3.8) we have considered convolution of a finite mixture of gamma densities. Instead of taking convolution of mixture of gamma densities , one can look into the problem of taking the convolution of gamma densities. That is

$$f = f_1 * f_2 * \ldots * f_m \qquad \text{--------------} \ (3.33)$$

where

$f_i(x) = \frac{{\theta_i}^{p_i}}{\Gamma(p_i)} e^{-\theta_i x} x^{p_i - 1}$, $\theta_i > 0$, $p_i \geq 1$, $i = 1,2, \ldots , m$, $x > 0$ and

$n = \sum_{i=1}^{m} p_i$.

In this set up Theorem(3.8) reads as follows.

The probability density function f of a non negative continuous random variable is of the form (3.33) if and only if its Laplace transform is of the form

$$\phi_f(s) = \frac{1}{\prod_{i=1}^{m}(1+\frac{s}{\theta_i})^{p_i}}$$

where θ_i , $p_i > 0$, $i = 1,2, \ldots , m$, and $n = \sum_{i=1}^{m} p_i$.

Since the class of finite convolutions of exponential densities is subset of the class of finite convolutions of gamma densities, the class of probability density functions having Laplace transforms with negative zeros and poles contains convolutions of exponential as well as convolutions of gamma densities.

CHAPTER IV

PROBABILITY DISTRIBUTIONS HAVING GENERATING FUNCTIONS WITH ZEROS AND POSITIVE POLES

4.1. A class of probability mass functions having probability generating functionwith zeros and positive poles.

As mentioned in section 2.3 when we turn our attention to the discrete case the probability generating function can be advantageously used instead of the Laplace transform to examine the distributional aspects. We begin the study by defining a class of distributions for which the generating function admits a useful representation.

Following the terminology in Srivastava and Manocha (1984), the function $g_f(s)$ which possesses a formal power series expansion in s such that

$g_f(s) = \sum_{n=0}^{\infty} f_n(x)\ s^n$

where each member of the set $\{ f_n(x) \}_{n=0}^{\infty}$ is independent of s is called a generating function for the set $\{f_n(x)\}$. If for some set of values of x, $g_f(s)$ is analytic at s=0, then $g_f(s)$ defined above converges in some domine of the complex plain that includes the origin.

We denote by E the class of real functions with domain {0,1,2, …} and generating function $g_f(s)$ with complex zeros and positive poles. That is

$$E = \left\{ f \mid g_f(s) = c_n \frac{\prod_{i=1}^{m}(1- b_i\, s)}{\prod_{j=1}^{n}(1- q_j\, s)} \right\} \qquad \text{------------} \ (4.1)$$

where c_n is a constant, $0 \le m < n < \infty$, $b_i \neq q_j$, $0 \le q_j \le 1$, j = 1,2, … , n and b_i's are complex for i = 1,2, … , m. Without loss of generality we assume that

$$0 \le q_1 \le q_2 \le \ \dots\ \le q_n < 1 \qquad \text{-----------------} \ (4.2)$$

Let E^+ denote the classof probability mass functions in E. That is

$$E^+ = \{ f \mid f \in E,\ f(x) \geq 0 \text{ and } \sum_{x=0}^{\infty} f(x) = 1 \}.$$

It can be noticed that the class E+ is closed under convolution and mixing . It may be further observed that where as in chapter III we dealt with Laplace transforms having negative poles, in the present chapter we consider situations where the probability generating function admits positive poles. The reason for this may be attributed to the fact that for the members of the class, which we discuss in the sequel, the poles should be positive.

We first consider the special case when the zeros in (4.1) are real and positive. In this case the class is closed under convolution but not under mixing . In fact if

$$f = f_1 * f_2 * \ldots * f_k$$

where $f_1, f_2, \ldots, f_k$ are independent probability mass functions having generating functions

$$g_{f_j}(s) = c_n \frac{\prod_{i=1}^{m}(1- b_{ij}\, s)}{\prod_{l=1}^{n}(1- q_{lj}\, s)} \quad j = 1,2, \ldots , k.$$

then

$$g_{f^*}(s) = \prod_{j=1}^{k} g_{f_j}(s)$$

$$= \prod_{j=1}^{k}\left[\ c_n \frac{\prod_{i=1}^{m}(1- b_{ij}\, s)}{\prod_{l=1}^{n}(1- q_{lj}\, s)}\ \right]$$

which is of the same form as (4.1) with b_i and q_j positive.

The latter part of the statement follows from the following counter example.

If

$$g_{f_1}(s) = \frac{11(1- \frac{3}{11}\, s)}{15\,(1- \frac{s}{3})(1- \frac{s}{5})}$$

$$= \frac{1}{2}\frac{\frac{2}{3}}{1-\frac{s}{3}} + \frac{1}{2}\frac{\frac{4}{5}}{1-\frac{s}{5}} ,$$

Then,

$$f_1(x) = \frac{1}{2}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^x + \frac{1}{2}\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^x , x = 0,1,2, \dots .$$

Obviously $f_1(x) \in E^+$ with b_i's positive and real.

Also if

If

$$g_{f_2}(s) = \frac{11(1-\frac{5}{11}s)}{18\,(1-\frac{s}{2})(1-\frac{s}{3})}$$

$$= \frac{1}{3}\frac{\frac{1}{2}}{1-\frac{s}{2}} + \frac{2}{3}\frac{\frac{2}{3}}{1-\frac{s}{3}} ,$$

Then,

$$f_2(x) = \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^x + \frac{2}{3}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^x , x = 0,1,2, \dots$$

Here also $f_2(x) \in E^+$ with b_i's positive and real.

Consider

$$g_f(s) = \frac{5}{11}g_{f_1}(s) + \frac{6}{11}g_{f_2}(s)$$

$$= \frac{2-\frac{157}{110}s+\frac{1}{11}s^2}{3\,(1-\frac{s}{2})(1-\frac{s}{3})(1-\frac{s}{5})}$$

Observing that $g_f(s)$ has only complex zeros, it follows that f does not belong to the class of probability mass functions having probability generating functions with positive zeros and positive poles.

We give below a sufficient condition for a real valued function f to have the representation(4.1) with real b_i's . This result can be viewed as the discrete version of the condition suggested in Steutel(1970), reviewed as Theorem 2.7.which is given in K.J. John and K. R. M. Nair (2000).

Theorem 4.1.

Let f be a real valued function with domain {0,1,2, …} for which the generating function has the representation (4.1) with m = n-1,

$$c_n = \left[\sum_{i=1}^{n} \frac{1}{p_i} \left\{ \frac{\prod_{j=1}^{n-1}(q_i - b_j)}{\prod_{j=1}^{n}(q_i - q_j)} \right\} \right]^{-1}$$

$0 \leq b_1 \leq b_2 \leq \ldots \leq b_{n-1} < \infty$

and ------------- (4.4)

$0 < q_1 \leq q_2 \leq \ldots \leq q_n < 1$

The condition

$q_i < b_i < q_{i-1}$, i = 1,2, … , (n-1) ---------------- (4.5)

implies that f is a probability mass function.

Proof :

Writing $g_f(s)$ in the form

$$g_f(s) = \frac{A_1}{1-q_1 s} + \frac{A_2}{1-q_2 s} + \ldots + \frac{A_n}{1-q_n s}. \quad \text{--------} \quad (4.6)$$

where A_i's are constants independent of s and using the representation (4.1) , we get

$c_n \prod_{i=1}^{n-1}(1- \ b_i s)= A_1 \prod_{j=2}^{n}(1-q_j s)+A_2 \prod_{\substack{j=1 \\ j\neq 2}}^{n}(1-q_j s)+\ldots+A_1 \prod_{j=1}^{n-1}(1-q_j s)$

----------------------------------- (4.7)

setting $s = \frac{1}{q_1}$ in (4.7) we get

$$A_1 = c_{n_1} \frac{\prod_{i=1}^{n-1}(1-\frac{b_1}{q_1})}{\prod_{j=1}^{n}(1-\frac{q_j}{q_1})}$$

A_1 is always positive since the numerator and denominator contains the same number of terms, each term on the numerator being negative by virtue of the conditions (4.5) and each term on the denominator being negative by (4.4)

In general setting $s = \frac{1}{q_1}$, in (4.7) we get

$$A_i = c_{n.} \frac{\prod_{j=1}^{n-1}(1-\frac{b_j}{q_i})}{\prod_{\substack{j=1 \\ j\neq i}}^{n}(1-\frac{q_j}{q_i})} \qquad i = 1,2, \ldots , n \qquad \text{-------- (4.8)}$$

which is also positive since the (i-1)terms of the numerator and denominator are positive by the same argument, given above. Hence all A_i's are positive. Now from (4.6)

f(x) $= A_1 {q_1}^x + A_2 {q_2}^x + \ldots + A_n {q_n}^x$ -------------- (4.9)

$\sum_{i=1}^{n}(\frac{A_i}{p_i}) p_i {q_i}^x =$, x =0,1,2, …

But f(x) ≥ 0 and$\sum_{x=0}^{\infty} f(x) = \sum_{i=1}^{n}(\frac{A_i}{p_i})$, which is equal to unity by (4.8) and the choice of c_n in the theorem.

Mixture of negative binomial distributions.

Analogous to the continuous case, in this section we identify certain classes of probability mass functions in E^+ which have practicalapplications.

A finite mixture of geometric distributions with mixing probabilities a_j , j = 1,2, … , n is defined by the probability mass function

$$f(x) = \sum_{j=1}^{n} a_j \, p_j {q_j}^x \qquad \text{------------------ (4.10)}$$

x = 0,1,2, … , $0 \le p_j \le 1$, $q_j = 1 - p_j$, $a_j > 0$, and $\sum_{j=1}^{n} a_j = 1$.

We give below a characterization theorem for the mixture of geometric distribution (4.10) based on a representation of probability generating function, which is given in K.J. John and K. R. M. Nair (2000).

Theorem 4.2

The probability mass function of a discrete random variable is specified by (4.10) if and only if the probability generating function has the representation

$$g_f(s) = c_n . \frac{\prod_{i=1}^{n-1}(1-b_i s)}{\prod_{j=1}^{n}(1-q_j s)} \qquad \text{---------------- (4.11)}$$

$0 < q_i < 1$ for all i =1,2, … , n, c_n is a constant and ${b_i}^{-1}$ for i =1,2, … , n-1 are the roots of the equation

$$\sum_{i=1}^{n} a_i \, p_i \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - q_j s) = 0 \qquad \text{------------------ (4.12)}$$

where$p_i = 1 - q_i$, $a_i > 0$ and $\sum_{i=1}^{n} a_i = 1$.

Proof :

When X has the probability mass function (4.10) , the probability generating function simplifies to

$$g_f(s) = \frac{a_1 p_1}{1-q_1 s} + \frac{a_2 p_2}{1-q_2 s} + \ldots + \frac{a_n p_n}{1-q_n s}$$

$$= \frac{a_1 p_1 (1-q_2 s)(1-q_3 s)\ldots(1-q_n s) + \cdots + a_n p_n (1-q_1 s)(1-q_2 s)\ldots(1-q_{n-1} s)}{\prod_{j=1}^{n}(1-q_j s)}$$

$$= \frac{\sum_{i=1}^{n} a_i p_i \prod_{\substack{j=1 \\ j\neq i}}^{n}(1-q_j s)}{\prod_{j=1}^{n}(1-q_j s)}$$

The above expression can be written using (4.12) as

$$g_f(s) = c_n . \frac{\prod_{i=1}^{n-1}(1-b_i s)}{\prod_{j=1}^{n}(1-q_j s)} \quad \text{----------------------} \quad (4.13)$$

Conversely assume that (4.11) holds. That is

$$g_f(s) = c_n . \frac{\prod_{i=1}^{n-1}(1-b_i s)}{\prod_{j=1}^{n}(1-q_j s)}$$

Inthe light of (4.12) we can write

$$g_f(s) = \frac{\sum_{i=1}^{n} a_i p_i \prod_{\substack{j=1 \\ j\neq i}}^{n}(1-q_j s)}{\prod_{j=1}^{n}(1-q_j s)}$$

$$= \frac{a_1 p_1 (1-q_2 s)(1-q_3 s)\ldots(1-q_n s) + \cdots + a_n p_n (1-q_1 s)(1-q_2 s)\ldots(1-q_{n-1} s)}{\prod_{j=1}^{n}(1-q_j s)}$$

$$= \frac{a_1 p_1}{1-q_1 s} + \frac{a_2 p_2}{1-q_2 s} + \ldots + \frac{a_n p_n}{1-q_n s}$$

This gives

f(x) = $\sum_{j=1}^{n} a_j\, p_j q_j{}^x$, x = 0,1,2, … , $0 \le p_j \le 1$, $q_j = 1 - p_j$,

$a_j > 0$, and $\sum_{j=1}^{n} a_j = 1$.

As an illustration, consider the probability generating function given by

$g_f(s) = \frac{18}{30} \frac{(1-b_1 s)(1-b_2 s)}{(1-\frac{3}{10}s)(1-\frac{4}{10}s)(1-\frac{5}{10}s)}$------------------(4.14)

where

${b_i}^{-1} = \frac{73-\sqrt{109}}{29}$

and

${b_i}^{-1} = \frac{73+\sqrt{109}}{29}$

$g_f(s)$ can be written as

$g_f(s) = \frac{1}{3}\frac{\frac{7}{10}}{(1-\frac{3}{10}s)} + \frac{1}{3}\frac{\frac{6}{10}}{(1-\frac{4}{10}s)} + \frac{1}{3}\frac{\frac{5}{10}}{(1-\frac{3}{10}s)}$

Hence

f(x) = $a_1(\frac{7}{10})(\frac{3}{10})^x + a_2(\frac{6}{10})(\frac{4}{10})^x + a_3(\frac{5}{10})(\frac{5}{10})^x$ -------------- (4.15)

with $a_1 = a_2 = a_3 = \frac{1}{3}$ and x = 0,1,2, …

Analogous to the mixture of geometric distributions defined in (4.10) , we define a finite mixture of negative binomial distributions with mixing probabilities a_j , j = 1,2, … , n as follows.

A discrete random variable X with support {0,1,2, … } is said to follow a mixture of negative binomial distributions if the probability mass function is of the form

f(x) = $\sum_{j=1}^{n} a_j \binom{-j}{x} p^j (-q)^x$, x= 0,1,2, … --------------- (4.16)

$0 \leq p \leq 1$, $q = 1 - p$, $0 < n < \infty$, $a_j > 0$, $\sum_{j=1}^{n} a_j = 1$

and

$$\binom{-j}{x} = \frac{(-1)^x j(j+1)\dots(j+x-1)}{x!}$$

Mixture distributions of the form (4.16) arises naturally in connection with the distribution of geometric sums as follows.

Let $X_1, X_2, \dots, X_N$ be independent and identically distributed non degenerate random variable with

$$P(X_i = x) = pq^x, \quad x = 0,1,2, \dots, 0 \le p \le 1, \quad q = 1 - p.$$

Assume that N is an integer valued random variable independent of the Xi and having probability mass function

$P(N=r) = a_r, \quad r = 1,2, \dots, n.$

Define $S_N = X_1 + X_2 + \dots + X_N$. Then the probability generating function of S_N is given by

$$g_{S_N}(s) = E(s^{S_N})$$

$$= \sum_{i=1}^{n} E[s^{S_N}/N = r]P[N=r]$$

$$= \frac{a_1 p}{1-qs} + \frac{a_2 p^2}{(1-qs)^2} + \frac{a_3 p^3}{(1-qs)^3} \dots + \frac{a_n p^n}{(1-qs)^n} \quad \text{---} \quad (4.17)$$

Since (4.17) is the probability generating function of (4.16), S_N follows a finite mixture of negative binomial distributions

The following theorem provides a characterization of the class of mixture of negative binomial distributions.

Theorem 4.3 .

Let X be a discrete random variable with support the set of non negative integers and having probability mass function f(x). Then X is distributed as a mixture of negative binomial distributions specified by (4.16) if and only if its probability generating function has the form

g_f(s) = $c_{n.} \frac{\prod_{i=1}^{n-1}(1-b_i s)}{(1-qs)^n}$-------------- (4.18)

1 < q < 1, c_n is a constant , and ${b_i}^{-1}$, for I = 1,2, … , n-1 are the roots of the equation

$\sum_{j=1}^{n} a_j p^j (1-qs)^{n-j} = 0$ -------------------- (4.19)

where p = 1 – q, a_j> 0, and $\sum_{j=1}^{n} a_j = 1$.

Proof :

When X has the distribution distribution (4.16) , its probability generating function simplifies to

g_f(s) = $\frac{a_1 p}{1-qs} + \frac{a_2 p^2}{(1-qs)^2} + \frac{a_3 p^3}{(1-qs)^3} \ldots + \frac{a_n p^n}{(1-qs)^n}$---- (4.20)

$$= \frac{\sum_{j=1}^{n} a_j p^j (1-qs)^{n-j}}{(1-qs)^n}$$

$$= c_{n.} \frac{\prod_{i=1}^{n-1}(1-b_i s)}{(1-qs)^n}$$

wherethe b_i's satisfies the relation (4.19).

Conversely assume that g_f(s) has the form (4.18). Using (4.19) we can write

$$g_f(s) = \frac{\sum_{j=1}^{n} a_j p^j (1-qs)^{n-j}}{(1-qs)^n}$$

$$= \frac{a_1 p \cdot (1-qs)^{n-1} + a_2 p^2 (1-qs)^{n-2} + \cdots + a_n p^n}{(1-qs)^n}$$

$$= \frac{a_1 p}{1-qs} + \frac{a_2 p^2}{(1-qs)^2} + \frac{a_3 p^3}{(1-qs)^3} \ldots + \frac{a_n p^n}{(1-qs)^n}$$

so that

f(x) = $\sum_{j=1}^{n} a_j \binom{-j}{x} p^j (-q)^x$, x= 0,1,2, …

$0 \le p \le 1$, $q = 1 - p$, $0 < n < \infty$, $a_j > 0$, $\sum_{j=1}^{n} a_j = 1$

The mixture of negative binomial distribution (4.16) can be generalized by incorporating an additional parameter as follows.

The probability mass function under present condition is

f(x) = $\sum_{j=1}^{m} a_j \binom{-j}{x} p^j (-q)^x$, x= 0,1,2, … --------------- (4.21)

$0 \le p \le 1$, $q = 1 - p$, $0 < n < \infty$, $a_j > 0$, $\sum_{j=1}^{m} a_j = 1$, m = n-k for k = 0,1,2, … , (n-1) and n is an integer.

The following Theorem provides a characterization of (4.21).

Theorem 4.4 .

A discrete random variable X in the support of {0,1,2, … } has probability mass function specified by (4.21) if and only if the probability generating function has the representation

g_f(s) = $c_m . \frac{\prod_{i=1}^{m-1}(1-b_i s)}{(1-qs)^n}$, bi ≠ 0 -----------(4.22)

$0 \le q \le 1$,c_m is a constant and ${b_i}^{-1}$, for i = 1,2, … , m-1 are the roots of the equation

$$\sum_{j=1}^{m} a_j p^j (1 - qs)^{m-j} = 0$$

with $p = 1 - q$, $a_j > 0$ and $\sum_{j=1}^{m} a_j = 1$.

The theorem follows by proceeding on the same lines of proof as that of Theorem 4.3.

We give below an example of a mixture of negative binomial distribution having probability generating function with a positive multiple zero and a positive multiple pole , by way of illustration.

For the negative binomial mixture specified by

$$f(x) = a_1\binom{-1}{x}\left(\tfrac{1}{3}\right)^{1}\left(-\tfrac{2}{3}\right)^{x} + a_2\binom{-2}{x}\left(\tfrac{1}{3}\right)^{2}\left(-\tfrac{2}{3}\right)^{x} + a_3\binom{-3}{x}\left(\tfrac{1}{3}\right)^{3}\left(-\tfrac{2}{3}\right)^{x},$$

x = 0,1,2, … , when $a_1 = a_3 = \frac{1}{4}$ and $a_2 = \frac{1}{2}$, the probability generating function is given by

$$g_f(s) = \frac{1}{4}\frac{\frac{1}{3}}{(1-\frac{2}{3}s)} + \frac{1}{2}\frac{(\frac{1}{3})^2}{(1-\frac{2}{3}s)^2} + \frac{1}{4}\frac{(\frac{1}{3})^3}{(1-\frac{2}{3}s)^3}$$

$$= \frac{4}{27}\frac{(1-\frac{s}{2})^2}{(1-\frac{2}{3}s)^3}$$

Here b_1^{-1} = +2 is a zero of order two and $q_1^{-1} = \frac{3}{2}$ is a pole of order three.

The following is an example of a mixture of a negative binomial distribution with single positive zeros and a multiple positive pole. For the negative binomial mixture specified by

$$f(x) = a_1\binom{-1}{x}\left(\tfrac{1}{2}\right)^{1}\left(-\tfrac{1}{2}\right)^{x} + a_2\binom{-2}{x}\left(\tfrac{1}{2}\right)^{2}\left(-\tfrac{1}{2}\right)^{x} + a_3\binom{-3}{x}\left(\tfrac{1}{2}\right)^{3}\left(-\tfrac{1}{2}\right)^{x}$$

, x=0,1,2, …, when $a_1 = a_3 = \frac{1}{6}$ and $a_2 = \frac{2}{3}$, the probability generating function is given by

$$g_f(s) = \frac{13}{48}\frac{(1-b_1 s)(1-b_2 s)}{(1-\frac{2}{3}s)^3}$$

The zeros are given by

$b_1^{-1} = 4 - \sqrt{3}$

and

$b_2{}^{-1} = 4 + \sqrt{3}$

and $q_1{}^{-1} = \frac{3}{2}$ is a multiple pole of order 3.

The generalization of the mixture of negative binomial in the form (4.21) admits only one multiple pole in the probability generating function. Yet another generalization can be formulated in which the probability generating function admits several multiple poles.

Consider the probability mass function given by

$$f(x) = \sum_{i=1}^{2} . \sum_{j=1}^{n_i} a_{ij} \binom{-j}{x} p_i{}^j (-q_i)^x, \quad x = 0,1,2, \ldots \qquad \text{--------------} \quad (4.23)$$

$0 \le p_i \le 1$, $q_i = 1 - p_i$, $0 < n_i < \infty$, for i = 1,2, $a_{ij} > 0$, $\sum_{i=1}^{2} . \sum_{j=1}^{n_i} a_{ij} = 1$.

The following theorem provides a characterization of (4.23)

Theorem 4.5 .

The probability mass function f of a non negative random variable with support {0,1,2, … } has the form (4.23) if and only if its probability generating function has the representation

$$g_f(s) = c_n . \frac{\prod_{i=1}^{n-1}(1-b_i s)}{(1-q_1 s)^{n_1}(1-q_2 s)^{n_2}} \quad , \quad bi \neq 0 \quad \text{-----------}(4.24)$$

where n_1 and n_2 are positive integers , n = $n_1 + n_2$, $0 \le q_i \le 1$ for i = 1,2, c_n is a constant and $b_i{}^{-1}$, for i = 1,2, … , n-1 are the roots of the equation

$$(1 - q_2 s)^{n_2} \sum_{i=1}^{n_1} a_{1i} p_1{}^i (1 - q_1 s)^{n_1 - i}$$

$$+(1 - q_1 s)^{n_1} \sum_{j=1}^{n_2} a_{2j} p_2{}^j (1 - q_2 s)^{n_2 - j} = 0.$$

with $0 \le p_i \le 1$, $q_i = 1 - p_i$ for all i =1,2, $a_{ij} > 0$ and $\sum_{i=1}^{2} . \sum_{j=1}^{n_i} a_{ij} = 1$.

Proof :

When X has probability mass function (4.23), its probability generating function is given by

$$g_f(s) = \frac{a_{11}p_1}{1-q_1 s} + \frac{a_{12}{p_1}^2}{(1-q_1 s)^2} + \ldots + \frac{a_{1n_1}{p_1}^{n_1}}{(1-q_1 s)^{n_1}}$$

$$+ \frac{a_{21}p_2}{1-q_2 s} + \frac{a_{22}{p_2}^2}{(1-q_2 s)^2} + \ldots + \frac{a_{2n_2}{p_2}^{n_2}}{(1-q_2 s)^{n_2}}$$

$$= \frac{(1-q_2 s)^{n_2} \sum_{i=1}^{n_1} a_{1i}{p_1}^{i}(1-q_1 s)^{n_1-i} + (1-q_1 s)^{n_1} \sum_{j=1}^{n_2} a_{2j}{p_2}^{j}\,(1-q_2 s)^{n_2-j}}{(1-q_1 s)^{n_1}(1-q_2 s)^{n_2}}$$

---------------- (4.6)

(4.26) can be written using (4.25) as

$$g_f(s) = c_n . \frac{\prod_{i=1}^{n-1}(1-b_i s)}{(1-q_1 s)^{n_1}(1-q_2 s)^{n_2}}$$

as claimed.

Conversely when $g_f(s)$ has the form (4.24),

$$g_f(s) = \frac{(1-q_2 s)^{n_2} \sum_{i=1}^{n_1} a_{1i}{p_1}^{i}(1-q_1 s)^{n_1-i} + (1-q_1 s)^{n_1} \sum_{j=1}^{n_2} a_{2j}{p_2}^{j}\,(1-q_2 s)^{n_2-j}}{(1-q_1 s)^{n_1}(1-q_2 s)^{n_2}}$$

$$= \frac{a_{11}p_1}{1-q_1 s} + \frac{a_{12}{p_1}^2}{(1-q_1 s)^2} + \ldots + \frac{a_{1n_1}{p_1}^{n_1}}{(1-q_1 s)^{n_1}}$$

$$+ \frac{a_{21}p_2}{1-q_2 s} + \frac{a_{22}{p_2}^2}{(1-q_2 s)^2} + \ldots + \frac{a_{2n_2}{p_2}^{n_2}}{(1-q_2 s)^{n_2}}$$

Hence

$$f(x) = \sum_{i=1}^{2} . \sum_{j=1}^{n_i} a_{ij} \binom{-j}{x} {p_i}^{j} (-q_i)^x, \quad x = 0,1,2, \ldots$$

$0 \le p_i \le 1$, $q_i = 1 - p_i$, $0 < n_i < \infty$, for i = 1,2, $a_{ij} > 0$,

$\sum_{i=1}^{2} . \sum_{j=1}^{n_i} a_{ij} = 1$.

The following theorem characterizes (4.27)

Theorem 4.6.

The probability mass function of a non negative discrete random variable X with support {0,1,2, …} has the form (4.27) if and only if its probability generating function has the form

$$g_f(s) = c_n . \frac{\prod_{i=1}^{n-1}(1-b_i s)}{\prod_{l=1}^{k} .(1-q_l s)^{n_l}} \quad , \qquad \text{-----------(4.28)}$$

where n_1 , $n_2 , \ldots , n_{k,}$are positive integers such that n = $\sum_{i=1}^{k} n_i$, , $0 \le q_i \le 1$ for i = 1,2,…, k, c_n is a constant and ${b_i}^{-1}$, for i = 1,2, … , n-1 are the roots of the equation

$$\sum_{i=1}^{k} . \sum_{j=1}^{n_i} a_{ij}\, {p_i}^{j} (1 - q_i s)^{n_i - j} \prod_{\substack{z=1 \\ z \ne i}}^{k} (1 - q_z s)^{n_z} = 0$$

with $0 \le p_i \le 1$, q_i = 1- p_i for all i =1,2, …, k,$a_{ij} > 0$ and $\sum_{i=1}^{k} . \sum_{j=1}^{n_i} a_{ij} = 1$.

The theorem follows proceeding along the same line of proof for Theorem 4.5.

Properties and characterization of mixture of geometric and negative binomial distribution were the theme of investigation in Section 4.2. Now we examine the behavior of the convolution of geometric and negative binomial distributions.

A finite convolution of geometric distribution is defined by the probability mass function

$$f = f_1 * f_2 * \ldots * f_n \text{------------------------} \quad (4.29)$$

where $f_i(x) = p_i q_i^{x}$, x = 0,1,2, …, $p_i > 0$, $q_i = 1-p_i$, i=1,2, … , n. The probability generating function of the convolution given in (4.29) is

$g_f(s) = \prod_{i=1}^{n} \left(\frac{p}{1-q_i s)} \right)$ ------------------ (4.30)

A finite convolution of negative binomial distribution is defined as

$f = f_1 * f_2 * \ldots * f_n$

where $f_i(x) = \binom{-r_i}{x} {p_i}^{r_i} (-q_i)^n$, $x = 0,1,2, \ldots$, $o \le p_i \le 1$, $q_i = 1-p_i$, $i=1,2, \ldots, n$ and in this case the probability generating function takes the form

$g_f(s) = \prod_{i=1}^{n} \left(\frac{p}{1-q_i s)} \right)^{r_i}$ ------------------------ (4.31)

It may be noticed from (4.30) and (4.31) that the finite convolution of geometric as well as negative binomial distributions belong to the class E^+.

CHAPTER V

PROPERTIES OF DISTRIBUTIONS WITH ZEROS AND POLES IN THE LAPLACE TRANSFORM / GENERATING FUNCTION

5.1. Introduction

The class of distributions encountered in the previous chapters possesses some interesting properties which are of use from the application point of view. In this chapter we give a brief discussion of the distributional aspects in respect of the class C^+ and E^+ followed by certain properties such as infinite divisibility and properties that arise in the context of geometric compounding . We begin with the properties of gamma mixture considered in section 3.2.

5.2. Moments and other properties of a finite gamma mixture.

For the mixture of gamma distribution defined by (3.8) the k^{th} order raw moment simplifies to

$$\mu_k^| = E(X^k)$$

$$= \sum_{j=1}^{n} a_j \frac{\theta^{n+1-j}}{\Gamma(n+1-j)} \int_0^\infty e^{-\theta x} x^{n+k-j} \, dx$$

$$= \sum_{j=1}^{n} a_j \frac{\theta^{n+1-j}}{\Gamma(n+1-j)} \frac{\Gamma(n+k-j+1)}{\theta^{n+k-j+1}}$$

$$= \frac{1}{\theta^k} \sum_{j=1}^{n} a_j \, (n+1-j)_k , \qquad \text{-------------------- (5.2)}$$

where $(n)_r = n(n+1)(n+2) \ldots (n+r-1)$ is the ascending factorial of order r.

In particular the mean and the variance of the distribution are given by

$$E(X) = \frac{1}{\theta}\sum_{j=1}^{n} a_j \,(n+1-j)_1 \quad \text{---------------------} \quad (5.2)$$

and

$$V(X) = \frac{1}{\theta^2}\sum_{j=1}^{n} a_j \,(n+1-j)(n+2-j) \; - \; [\frac{1}{\theta}\sum_{j=1}^{n} a_j \,(n+1-j)]^2 \text{ --- } (5.3)$$

Expressions for the mean and variance can also be obtained by using the characteristic function.

The characteristic function of (3.8) simplifies to

$$\phi(t) = \frac{a_1}{(1-\frac{it}{\theta})^n} + \frac{a_2}{(1-\frac{it}{\theta})^{n-1}} + \ldots + \frac{a_m}{(1-\frac{it}{\theta})}.$$

Using the terminology used in chapter III, in terms of the zeros and poles in the Laplace transform , we can write the characteristic function as

$$\phi(t) = \frac{\prod_{j=1}^{n-1}(1-\frac{it}{\eta_j})}{(1-\frac{it}{\theta})^n} \quad \text{----------} \quad (5.4)$$

where η_i's are the roots of the polynomial equation

$$\sum_{j=1}^{n} a_j \,(1+\frac{s}{\theta})^{j-1} = 0. \quad \text{-----------} \quad (5.5)$$

$a_j \geq 0$ and $\sum_{j=1}^{n} a_j = 1$.

Thecumulant generating function is given by

$k(t) = \log \phi(t)$

$$= \sum_{j=1}^{n-1} log\,(1-\frac{it}{\eta_j}) - n \log (1-\frac{it}{\theta}) \quad \text{------------} \quad (5.6)$$

In the region | t | <θ, log ϕ(t) admits Maclaurins expansion, so that (5.6) can be expressed as

$$k(t) = \sum_{j=1}^{n-1}\{-\frac{it}{\eta_j} - \frac{(it)^2}{2\,\eta_j{}^2} - \frac{(it)^3}{3\,\eta_j{}^3} - \frac{(it)^4}{4\,\eta_j{}^4} - \ldots\} + n\,\{\frac{it}{\theta} + \frac{(it)^2}{2\,\theta^2} + \ldots\}.$$

It follows that

$k_1 \quad = \frac{n}{\theta} - \sum_{j=1}^{n-1} \frac{1}{\eta_j}$

$k_2 \quad = \frac{n}{\theta^2} - \sum_{j=1}^{n-1} \frac{1}{\eta_j^2}$ and

$k_3 \quad = \frac{2n}{\theta^3} - 2\sum_{j=1}^{n-1} \frac{1}{\eta_j^3}$ --------------- (5.7)

$k_4 \quad = 6 \{\frac{n}{\theta^4} - \sum_{j=1}^{n-1} \frac{1}{\eta_j^4}\}$

Using the relation between cumulants and central moments, we have for the gamma mixture

Mean = $\frac{n}{\theta} - \sum_{j=1}^{n-1} \frac{1}{\eta_j}$ -------------- (5.8)

and

Variance = $\frac{n}{\theta^2} - \sum_{j=1}^{n-1} \frac{1}{\eta_j^2}$ ------------(5.9)

Further,

Skewness $= \frac{\mu_3^2}{\mu_2^3}$

$$= \frac{\{\frac{2n}{\theta^3} - 2\sum_{j=1}^{n-1} \frac{1}{\eta_j^3}\}^2}{\{\frac{n}{\theta^2} - \sum_{j=1}^{n-1} \frac{1}{\eta_j^2}\}^3}$$ ------------ (5.10)

and

Kurtosis $= \beta_2$

$$= \frac{\mu_4}{\mu_2^2}$$

$$= 3 + \frac{6\{\frac{n}{\theta^4} - \sum_{j=1}^{n-1} \frac{1}{\eta_j^4}\}}{\{\frac{n}{\theta^2} - \sum_{j=1}^{n-1} \frac{1}{\eta_j^2}\}^2}$$

It may be observed from (5.10) and (5.11) that as n becomes large the distribution tends to be symmetric and mesokurtic.

We can also determine the zeros of Laplace transforms of (3.8) in terms of the mixing probabilities a_j . We consider the simple case when n=3. In this situation, using (5.7), we have

$$k_1 = \frac{3}{\theta} - \sum_{j=1}^{2} \frac{1}{\eta_j} \text{------------------} \quad (5.12)$$

and

$$k_2 = \frac{3}{\theta^2} - \sum_{j=1}^{2} \frac{1}{{\eta_j}^2} \text{------------} \quad (5.13)$$

Also from (5.2) and (5.3) we have, when n = 3,

$$\mu_1^I = \frac{1}{\theta} \sum_{j=1}^{3} a_j (4\text{-}j)$$

$$= \frac{3a_1+2a_2+a_3}{\theta} \text{----------------} \quad (5.14)$$

and

$$\mu_2 = \frac{1}{\theta^2} \sum_{j=1}^{3} a_j \,(4\text{-}j)(5\text{-}j) - \left\{ \frac{1}{\theta} \sum_{j=1}^{3} a_j (4-j) \right\}^2 \text{----}(5.15)$$

Using the relations(5.12), (5.13) and (5.15) in the relations $k_1 = \mu_1^1$ and $k_2 = \mu_2$ we get

$$\sum_{j=1}^{2} \frac{1}{\eta_j} = \frac{1}{\theta} \{3\text{-}(3a_1 + 2a_2 + a_3)\} \quad \text{--------------} \ (5.16)$$

and

$$\sum_{j=1}^{2} \frac{1}{{\eta_j}^2} = \frac{1}{\theta^2} \{3\text{-}(12a_1 + 6a_2 + 2a_3) + (3a_1 + 2a_2 + a_3)\}$$

$$\text{--------------------} \ (5.17)$$

The values of $_1$ and η_2 can be obtained as the solution of equations (5.16) and (5.17).

Graphs of (3.8) for different values of n, mixing probabilities $\frac{1}{n}$ each and $\theta = 1$ are given in Fig. 5.1 and that for n=4 and $\theta = 1$ for different values of the mixing probabilities are exhibited in Fig. 5.2.

5.3. Infinite divisibility of the class C^+ having Laplace transforms with negative zeros and poles.

We presently establish that the class of distributions in C+ having Laplace transforms with negative zeros and poles is infinitely divisible under certain conditions. The general form of Laplace transform of density functions having negative zeros and poles is

$$w(s) = \frac{\prod_{i=1}^{m}(1+\frac{s}{\eta_i})}{\prod_{j=1}^{k}(1+\frac{s}{\theta_j})^{n_j}} \qquad \text{----------} \quad (5.18)$$

where $0 < \eta_1 \leq \eta_2 \leq \ldots \leq \eta_m < \infty$ and $0 < \theta_1 < \theta_2 < \cdots < \theta_{n_k} < \infty$.

The following theorem concerns the infinite divisibility of the class C^+ .

Theorem 5.1.

The function w(s) defined in (5.18) is the Laplace transform of an infinitely divisible distribution if

$$\frac{\sum_{j=1}^{k} n_j e^{-\theta_j x}}{\sum_{i=1}^{m} e^{-\eta_i x}} \geq 1. \qquad \text{------------------} \ (5.19)$$

Proof:

From (5.18) we have $w(s) = e^{-\Psi(s)}$ where $\boldsymbol{\Psi}(s)$ is given by

$$\boldsymbol{\Psi}(s) = \int_0^\infty \left(\frac{1-e^{-sx}}{x}\right)\left(\sum_{j=1}^{k} n_j e^{-\theta_j x} - \sum_{i=1}^{m} e^{-\eta_i x}\right)dx$$

From (5.19) we have

$$\left(\sum_{j=1}^{k} n_j e^{-\theta_j x} - \sum_{i=1}^{m} e^{-\eta_i x}\right) \geq 0.$$

Using the result from (2.10) to (2.12) we see that w(s) corresponds to the Laplace transform of an infinitely divisible distribution**.**

5.4. Gamma mixture in the context of geometric compounding.

As pointed out in section (3.2), gamma mixture arises naturally in the context of geometric compounding. It was established that the distribution of the N^{th} partial sum, S_N, follows the gamma mixture with mixing probabilities

$P(N=j) = a_{n+1-j}$, j=1,2, … , n.

when each of the X_i's follows the exponential law.

We presently prove a theorem concerning the form of the distribution of S_N.

Theorem 5.2.

LetX1, X2, … , X_N be independent and identically distributed random variables and S_N = X1 + X2 + … + $X_N$where N is an integer valued random variable . Then any two of the following statements implies the third.

(i) S_N has a mixture gamma distribution as defined in (3.8).
(ii) $P(N=r) = a_{n+1-r}$ for all r = 1,2, … , n.
(iii) X_i, i= 1,2, … , N is distributed as exponential with θ^{-1}.

Proof:

When conditions (ii) and (iii) holds, the distribution of SN turns out to be the gamma mixture (3.8), which is established in section (3.2) . Hence (i) holds.

Assume the (i) and (ii) holds. Observing that the Laplace transform of the mixture of gamma distribution defined by (3.8) is given by

$$\phi_{S_N}(s) = \frac{a_1}{(1+\frac{s}{\theta})^n} + \frac{a_2}{(1+\frac{s}{\theta})^{n-1}} + \ \dots \ + \frac{a_n}{(1+\frac{s}{\theta})}. \quad \text{------} \quad (5.20)$$

when (i) holds the Laplace transform of S_N takes the form (5.20) . But by definition , the Laplace transform of S_N is

$$\phi_{S_N}(s) = E(e^{-sS_N})$$

$= \sum_{r=1}^{n} E(e^{-sS_N}/N = r) P(N = r)$

$= \sum_{r=1}^{n} E(e^{-s(X_1+X_2+\cdots+X_r)}) P(N = r)$

$= a_1 Ee^{-s(X_1+X_2+\cdots+X_n)} + a_2 Ee^{-s(X_1+X_2+\cdots+X_{n-1})}$

$+ \ldots + a_n Ee^{-sX_1}$ ----------------------- (5.21)

From (5,20) and (5.21) we get

$$Ee^{-sX_1} = \frac{1}{(1+\frac{s}{\theta})}.$$

This implies that Xi is distributed as exponential with parameter $1/\theta$. Also from (5.20) and (5.21) we have

$$Ee^{-s(X_1+X_2)} = \frac{1}{(1+\frac{s}{\theta})^2}$$

That is

$$Ee^{-sX_1}Ee^{-sX_2} = \frac{1}{(1+\frac{s}{\theta})^2}$$

Hence X_2 follows exponential distribution with parameter $1/\theta$.

Proceeding like this we see that X_i is distributed as exponential with parameter $1/\theta$, establishing (iii).

Now assume that (i) and (iii) holds. The Laplace transform of X_i is

$$\phi_{X_i}(s) = \frac{1}{(1+\frac{s}{\theta})}.$$

The Laplace transform of S_N is

$$\phi_{S_N}(s) = E(e^{-sS_N})$$

$= \sum_{r=1}^{n} E(e^{-sS_N}/N = r) P(N = r)$

$$= \frac{P(N=1)}{(1+\frac{s}{\theta})} + \frac{P(N=2)}{(1+\frac{s}{\theta})^2} + \ldots + \frac{P(N=n)}{(1+\frac{s}{\theta})^n} \quad \text{--------- (5.22)}$$

Equating the coefficient of $(1+\frac{s}{\theta})^r$, r = 1,2, … , n in the expression for $\phi_{S_N}(s)$given by (5.22) and (5.20) we get

P(N=r) = a_{n+1-r} , r = 1,2, … , n

which is same as (ii).

5.5. Moments and other properties of mixture of negative binomial distribution.

The probability generating function of the mixture of negative binomial distribution defined by (4.16) is

$$g_f(s) = \frac{a_1 P}{(1-qs)} + \frac{a_2 p^2}{(1-qs)^2} + \ldots + \frac{a_n p^n}{(1-qs)^n}$$

we have

$$EX^{(r)} = E[X(X-1)(X-2) \ldots \{X-(r-1)\}$$

$$= \frac{d^r g_f(s)}{ds^r}|_{s=1} \qquad \text{-----------------------}(5.23)$$

It follows from (5.23) that

$$EX^{(1)} = E(X)$$

$$= \frac{q}{p}\sum_{j=1}^{n} j\, a_j$$

$$E(X)^{(2)} = \frac{q^2}{p^2}\sum_{j=1}^{n} j(j+1)a_j$$

Variance = $E(X)^{(2)} + E(X)^{(1)} - \{EX^{(1)}\}^2$

$$= \frac{q^2}{p^2}\sum_{j=1}^{n} j(j+1)a_j + \frac{q}{p}\sum_{j=1}^{n} j\, a_j - \{\frac{q}{p}\sum_{j=1}^{n} j\, a_j\}^2$$

Graphs of (4.16) for different values of n and mixing probabilities 1/n each are given in Fig. 5.3. to Fig. 5.6. and that for n=4 and p=1/3 for different values of the mixing probabilities are given in Fig. 5.7. and Fig. 5.8.

The probability generating function of the mixture of geometric distribution defined in (4.10) is

$$g_f(s) = \frac{a_1 p_1}{(1-q_1 s)} + \frac{a_2 p_2}{(1-q_2 s)^2} + \ldots + \frac{a_n p_n}{(1-q_n s)^n}$$

Hence

$$\text{Mean} = \sum_{j=1}^{n} a_j \frac{q_j}{p_j}$$

and

$$\text{Variance} = 2 \sum_{j=1}^{n} a_j \frac{{q_j}^2}{{p_j}^2} + \sum_{j=1}^{n} a_j \frac{q_j}{p_j} - \{\sum_{j=1}^{n} a_j \frac{q_j}{p_j}\}^2$$

Graphs of the mixture of geometric distribution defined in (4.10) for different values of mixing probabilities are given in Fig. 5.9 to Fig. 5.11

5.6 Negative binomial mixture in the context of geometric compounding.

In section 4.2 we have discussed the generation of negative binomial mixture by compounding. It was shown that the sum of n independent identically distributed random variables follows the mixture of negative binomial distributions with mixing probabilities

$P(N=j) = a_j$, $j= 1,2, \ldots , n$

when each of X_i's follows the geometric law.

The following theorem concerns the form of distribution of S_N.

Theorem 5.3.

Let $X_1, X_2, \ldots, X_N$ be independent and identically distributed random variables and

$S_N = X_1 + X_2 + \ldots + X_N$.

where N is an integer valued random variable. Then any two of the following statements implies the other.

(i) $S_N \overset{d}{=}$ NBMn

(ii) P(N=j) = a_j , j =1,2, … , n

(iii) $X_i \overset{d}{=}$ geometric (p).

NBMn is the mixture of negative binomial defined in (4.16).

Proof:

When conditions (ii) and (iii) holds , the validity of (i) is established in Section (4.2).

Assume that conditions (i) and (ii) holds. The probability generating function of the mixture of negative binomial distribution is

$$g_{S_N}(s) = \frac{a_1 P}{(1-qs)} + \frac{a_2 p^2}{(1-qs)^2} + \ldots + \frac{a_n p^n}{(1-qs)^n} \text{----------} (5.24)$$

when (i) and (ii) holds (5.24) takes the form

$$g_{S_N}(s) = \frac{P\,P(N=1)}{(1-qs)} + \frac{p^2 P(N=2)}{(1-qs)^2} + \ldots + \frac{p^n P(N=n)}{(1-qs)^n} \text{-----} (5.25)$$

The probability generating function of S_N also can be written as

$$\phi_{S_N}(s) = E(e^{-sS_N})$$

$$= \sum_{r=1}^{n} E(e^{-sS_N}/N = r)\, P(N = r)$$

$$= \sum_{r=1}^{n} E(e^{-s(X_1+X_2+\cdots+X_r)})\, P(N = r)$$

$$= P(N=1)\, E\, e^{-s\,X_1} + p(N=2)\, E\, e^{-s(X_1+X_2)} + \ldots$$

$$+ P(N=n)\, E\, e^{-s(X_1+X_2+X_3)} \text{-------------}(5.26)$$

From (5.25) and (5.26) we get

$$E\, e^{-s\,X_1} = \frac{p}{1-qs} \text{------------------}(5.27)$$

Therefore

$X_1 \overset{d}{=}$ geometric (p).

$E\, e^{-s(X_1+X_2)} = \dfrac{p^2}{(1-qs)^2}$ ------------- (5.28)

Since X1 and X2 are independent, from (5.27) and (5.28)

$X_2 \overset{d}{=}$ geometric (p).

Proceeding like this we can show that

$X_i \overset{d}{=}$ geometric (p).

Now assume that the conditions (i) and (iii) holds. Hence the probability generating function of S_N is given by

$$\phi_{S_N}(s) = E(e^{-sS_N})$$

$$= \sum_{r=1}^{n} E(e^{-sS_N}/N = r)\, P(N = r)$$

$$= \frac{PP(N=1)}{(1-qs)} + \frac{P^2P(N=2)}{(1-qs)^2} + \ldots + \frac{P^nP(N=n)}{(1-qs)^n} \text{----(5.29)}$$

From (5.24) and (5.29) , equating the coefficient of $\{\frac{p}{1-qs}\}^j$ we get

P(N=j) = aj , j=1,2, … ,n.

which is same as (ii).

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